D. K. Faddeev & I. S. Sominskii

Problems in Higher Algebra

Translated by J. L. Brenner

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Problems in Higher Algebra

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D. K. Faddeev, I. S. Sominskii

Translated by

J. L. BRENNER

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PREFACE

Selections from this fine collection of problems have been quoted ever since its first appearance in 1954. It is time to make it more widely available to the western world.

The section on linear equations will be interesting because of its systematic coverage, and because it uses the vector-space of the n-tuples of coefficients to expound the principal results. Calculation of determinants has been lifted from the sterile to the fruitful by means of imaginative problems, many of them solved by use of mathematical induction. At the same time, there are plenty of simple problems, but written by a master who can make the material interesting.

The section on Sturm sequences includes applications to the classical polynomials--the polynomials of Legendre, of Hermite, of Laguerre; and to the truncated exponential series. A feature of the book is that every problem, or at the most every group of two or three problems, is independent, and sections of the book can be sampled or omitted at the user's discretion.

I have added a few problems to the section on matrices; I learned them at the 1964 Gatlinburg conference. I hope the mathematical public will find these and other additions attractive.

This book is recommended as an adjunct text, as a problem book, and for self study. It is a pleasure to acknowledge the splendid cooperation of the publisher, and the consummate skill of Shirley DeMeo, the typist-compositor.

Joel Brenner

Menlo Park, California January, 1965

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CHAPTER I - PROBLEMS

COMPLEX NUMBERS

1. RECKONING WITH COMPLEX NUMBERS

1 If x and y are real numbers, and if the relation

$$(1 + 2i)\,x + (3 - 5i)\,y = 1 - 3i,$$

is satisfied, find the values of x and y.

2 If x, y, z, t are real numbers, solve the following system of equations:

$$(1+i)\,x+(1+2i)\,y+(1+3i)\,z+(1+4i)\,t=1+5i,$$
$$(3-i)\,x+(4-2i)\,y+(1+i)\,z+4it=2-i.$$

3 If n is a whole number, find the value of i^n.

4 Check the following identity

$$x^4+4=(x-1-i)(x-1+i)(x+1+i)(x+1-i).$$

5 Perform the following computations:

a) $(1+2i)^6$; b) $(2+i)^7+(2-i)^7$; c) $(1+2i)^5-(1-2i)^5$.

6 Under what conditions is the product of two complex numbers pure imaginary?

7 Calculate the following quantities:

a) $\frac{1+i\operatorname{tg}\alpha}{1-i\operatorname{tg}\alpha}$; b) $\frac{a+bi}{a-bi}$; c) $\frac{(1+2i)^2-(1-i)^3}{(3+2i)^3-(2+i)^2}$;

d) $\frac{(1-i)^5-1}{(1+i)^5+1}$; e) $\frac{(1+i)^9}{(1-i)^7}$.

8 If n is a rational integer, compute the value of the fraction $\frac{(1+i)^n}{(1-i)^{n-2}}$.

9 Solve the following three systems of equations:

a) $(3-i)x+(4+2i)y=2+6i,\quad (4+2i)x-(2+3i)y=5+4i;$

b) $(2+i)x+(2-i)y=6,\quad (3+2i)x+(3-2i)y=8;$

c) $x+yi-2z=10,\quad x-y+2iz=20,\quad ix+3iy-(1+i)z=30.$

10 Compute the following:

a) $\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2$; b) $\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^3$.

*11 If ω stands for $-\frac{1}{2}+\frac{i\sqrt{3}}{2}$, compute the following:

a) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$;

b) $(a+b)(a+b\omega)(a+b\omega^2)$;

c) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$;

d) $(a\omega^2+b\omega)(b\omega^2+a\omega)$.

12 Find a complex number which is conjugate to a) its square, b) its cube.

*13 Prove the following theorem. Suppose the numbers $x_1, x_2, \dots, x_n$ are combined in any rational way (that is, using addition, subtraction, multiplication, or division) and the result of this combination is u. Then the result of carrying out the same rational operations on the conjugate complex numbers $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ will be the number $\bar{u}$; that is a complex conjugate of u.

14 If $x + yi = (s + ti)^n$, then show that $x^2 + y^2 = (s^2 + t^2)^n$.

15 Compute the values of the following quantities:

a) $\sqrt{2i}$; b) $\sqrt{-8i}$; c) $\sqrt{3-4i}$; d) $\sqrt{-15+8i}$;
e) $\sqrt{-3-4i}$; f) $\sqrt{-11+60i}$; g) $\sqrt{-8+6i}$;
h) $\sqrt{-8-6i}$; i) $\sqrt{8-6i}$; j) $\sqrt{8+6i}$; k) $\sqrt{2-3i}$;
l) $\sqrt{4+i}+\sqrt{4-i}$; m) $\sqrt{1-i\sqrt{3}}$; n) $\sqrt[4]{-1}$;
o) $\sqrt[4]{2-i\sqrt{12}}$.

16 Given that $\sqrt{a+bi} = \pm(\alpha+\beta i)$, what are the values of $\sqrt{-a-bi}$?

17 Solve the following equations:

a) $x^2-(2+i)x+(-1+7i)=0$;
b) $x^2-(3-2i)x+(5-5i)=0$;
c) $(2+i)x^2-(5-i)x+(2-2i)=0$.

*18 Factor each of the following into factors with real coefficients and solve:

a) $x^4+6x^3+9x^2+100=0$; b) $x^4+2x^2-24x+72=0$.

19 Solve the equations:

a) $x^4-3x^2+4=0$; b) $x^4-30x^2+289=0$.

20 If $p^2/4 - q < 0$, find a formula for solving the bi-quadratic equation $x^4 + px^2 + q = 0$.

2. COMPLEX NUMBERS IN POLAR FORM

21 In the Argand diagram, locate the following complex numbers:

$1,\ -1,\ -\sqrt{2},\ i,\ -i,\ i\sqrt{2},\ -1+i,\ 2-3i.$

22 Find the polar form for the following complex numbers:

a) 1; b) -1; c) i; d) $-i$; e) $1+i$; f) $-1+i$; g) $-1-i$;
h) $1-i$; i) $1+i\sqrt{3}$; j) $-1+i\sqrt{3}$; k) $-1-i\sqrt{3}$;
l) $1-i\sqrt{3}$; m) $2i$; n) -3; o) $\sqrt{3}-i$; p) $2+\sqrt{3}+i$.

23 Use tables to calculate approximately the arguments of the following numbers and thus write them in polar form:

a) $3 + i$; b) $4 - i$; c) $-2 + i$; d) $-1 - 2i$.

24 Find the geometric locus of those points that satisfy the following conditions:

a) the modulus of the points is 1;

b) the points all have argument $\pi/6$.

25 Find the geometric locus in the z-plane of those points that satisfy the following conditions:

a) $|z| < 2$; b) $|z-i| \leqslant 1$; c) $|z-1-i| < 1$.

26 Solve the following equations:

a) $|x| - x = 1 + 2i$; b) $|x| + x = 2 + i$.

*27 Establish the following identity, valid for any two complex numbers x, y:

$$|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$$

Find a geometric interpretation of the above identity.

*28 Let z be any complex number except -1, $z \neq -1$. Show that there is a real number t, such that z can be written in the form $z = \dfrac{1 + ti}{1 - ti}$.

29 Find conditions under which the modulus of the sum of two complex numbers is equal to the difference of the moduli of the individual terms.

30 Find conditions under which the modulus of the sum of two complex numbers is equal to the sum of the moduli of the individual terms.

*31 Let $u = \sqrt{zz'}$, where z and z' are two complex numbers. Establish the following identity:

$$|z| + |z'| = \left|\frac{z+z'}{2} - u\right| + \left|\frac{z+z'}{2} + u\right|.$$

32 If $|z| < \frac{1}{2}$, then show that the following relationship is satisfied:

$$|(1+i)z^3 + iz| < \frac{3}{4}.$$

33 Establish the following:

$$(1+i\sqrt{3})(1+i)(\cos\varphi + i\sin\varphi) =$$
$$= 2\sqrt{2}\left[\cos\left(\frac{7\pi}{12}+\varphi\right) + i\sin\left(\frac{7\pi}{12}+\varphi\right)\right].$$

34 Simplify the fraction $\dfrac{\cos\varphi + i\sin\varphi}{\cos\psi - i\sin\psi}$.

35 Compute the following: $\dfrac{(1-i\sqrt{3})(\cos\varphi + i\sin\varphi)}{2(1-i)(\cos\varphi - i\sin\varphi)}$.

36 Compute the following quantities:

a) $(1+i)^{25}$; b) $\left(\dfrac{1+i\sqrt{3}}{1-i}\right)^{20}$; c) $\left(1 - \dfrac{\sqrt{3}-i}{2}\right)^{24}$;

d) $\dfrac{(-1+i\sqrt{3})^{15}}{(1-i)^{20}} + \dfrac{(-1-i\sqrt{3})^{15}}{(1+i)^{20}}$.

37 If n is a rational integer, show that the following relations hold:

a) $(1+i)^n = 2^{\frac{n}{2}}\left(\cos\dfrac{n\pi}{4} + i\sin\dfrac{n\pi}{4}\right)$;

b) $(\sqrt{3}-i)^n = 2^n\left(\cos\dfrac{n\pi}{6} - i\sin\dfrac{n\pi}{6}\right)$;

*38 Let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.

Calculate $(1+\omega)^n$.

39 Compute the value of $\omega_1^n + \omega_2^n$, where n is a rational integer and ω_1 , and ω_2 are defined by

$$\omega_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \omega_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2},$$

*40 Compute $(1 + \cos\alpha + i\sin\alpha)^n$.

*41 Given that $z + \frac{1}{z} = 2\cos\theta$, show that

$$z^m + \frac{1}{z^m} = 2\cos m\theta.$$

42 Establish the relation $\left(\frac{1+i\operatorname{tg}\alpha}{1-i\operatorname{tg}\alpha}\right)^n = \frac{1+i\operatorname{tg} n\alpha}{1-i\operatorname{tg} n\alpha}$.

43 Extract the following roots:

a) $\sqrt[3]{i}$; b) $\sqrt[3]{2-2i}$; c) $\sqrt[4]{-4}$; d) $\sqrt[6]{1}$; e) $\sqrt[6]{-27}$.

44 Compute the approximate numerical value of the following roots using trigonometric tables:

a) $\sqrt[3]{2+i}$; b) $\sqrt[3]{3-i}$; c) $\sqrt[5]{2+3i}$.

45 Compute the following:

a) $\sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$; b) $\sqrt[8]{\frac{1+i}{\sqrt{3}-i}}$; c) $\sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$.

46 If it is known that β is one of the values of $\sqrt[n]{\alpha}$, find all values of $\sqrt[n]{\alpha}$.

47 Express the following in terms of cos x, sin x:

a) cos 5x; b) cos 8x; c) sin 6x; d) sin 7x.

48 Express tan 6φ in terms of tan φ.

49 Establish formulas that express cos nx, sin nx, in terms of cos x, sin x.

50 Express the following as trigonometric polynomials (a trigonometric polynomial is a sum of monomial terms, each term being a constant multiple of the sine or cosine of an integral multiple of x):

a) $\sin^3 x$; b) $\sin^4 x$; c) $\cos^5 x$; d) $\cos^6 x$.

*51 Establish the following identities:

a) $2^{2m}\cos^{2m} x = 2\sum_{k=0}^{m-1} C_k^{2m}\cos 2(m-k)x + C_m^{2m}$;

b) $2^{2m}\cos^{2m+1} x = \sum_{k=0}^{m} C_k^{2m+1}\cos(2m-2k+1)x$

c) $2^{2m}\sin^{2m} x = 2\sum_{k=0}^{m-1}(-1)^{m+k}C_k^{2m}\cos 2(m-k)x + C_m^{2m}$

d) $2^{2m}\sin^{2m+1} x = \sum_{k=0}^{m}(-1)^{m+k}C_k^{2m+1}\sin(2m-2k+1)x.$

*52 Establish the following: $2\cos mx = (2\cos x)^m - \frac{m}{1}(2\cos x)^{m-2}$

$$+\frac{m(m-3)}{1\cdot 2}(2\cos x)^{m-4} - \dots$$

$$+(-1)^p\frac{m(m-p-1)(m-p-2)\dots(m-2p+1)}{p!}(2\cos x)^{m-2p} + \dots$$

*53 Express the fraction $\dfrac{\sin mx}{\sin x}$ in terms of cos x.

*54 Collapse the following sums:

a) $1 - C_2^n + C_4^n - C_6^n + \cdots;$

b) $C_1^n - C_3^n + C_5^n - C_7^n + \cdots$

*55 Establish the following identities:

$$\text{a)}\ 1 + C_4^n + C_8^n + \ldots = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}} \cos\frac{n\pi}{4}\right);$$

$$\text{b)}\ C_1^n + C_5^n + C_9^n + \ldots = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}} \sin\frac{n\pi}{4}\right);$$

$$\text{c)}\ C_2^n + C_6^n + C_{10}^n + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}} \cos\frac{n\pi}{4}\right);$$

$$\text{d)}\ C_3^n + C_7^n + C_{11}^n + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}} \sin\frac{n\pi}{4}\right).$$

*56 Compute the following sum:

$$C_1^n - \frac{1}{3} C_3^n + \frac{1}{9} C_5^n - \frac{1}{27} C_7^n + \ldots$$

57 Let n be the largest integral multiple of 3 that does not exceed m. Establish the following relation

$$(x + a)^m + (x + a\omega)^m + (x + a\omega^2)^m$$
$$= 3x^m + 3C_3^m\ {}^{m-3}a^3 + \ldots + 3C_n^m\ {}^{m-n}a^n,$$

where

$$\omega = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3},$$

58 Establish the following three relations:

$$\text{a)}\ 1 + C_3^n + C_6^n + \ldots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right);$$

$$\text{b)}\ C_1^n + C_4^n + C_7^n + \ldots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right);$$

$$\text{c)}\ C_2^n + C_5^n + C_8^n + \ldots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-4)\pi}{3}\right).$$

59 Compute the following sums:

a) $1+a\cos\varphi+a^2\cos 2\varphi+\ldots+a^k\cos k\varphi$;

b) $\sin\varphi+a\sin(\varphi+h)+a^2\sin(\varphi+2h)+\ldots+a^k\sin(\varphi+kh)$;

c) $\frac{1}{2}+\cos x+\cos 2x+\ldots+\cos nx$.

60 Establish the following:

$$\sin x+\sin 2x+\ldots+\sin nx=\frac{\sin\frac{n+1}{2}x\cdot\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

61 Compute the following limit:

$$\lim_{n\to\infty}\left(1+\frac{1}{2}\cos x+\frac{1}{4}\cos 2x+\ldots+\frac{1}{2^n}\cos nx\right).$$

62 If n is a positive integer, and θ is the angle that satisfies the relation $\sin\frac{\theta}{2}=\frac{1}{2n}$, show that the following relation is valid:

$$\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}+\ldots+\cos\frac{2n-1}{2}\theta=n\sin n\theta.$$

63 Establish each of the following:

a) $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$;

b) $\cos\frac{2\pi}{11}+\cos\frac{4\pi}{11}+\cos\frac{6\pi}{11}+\cos\frac{8\pi}{11}+\cos\frac{10\pi}{11}=-\frac{1}{2}$;

c) $\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{9\pi}{13}+\cos\frac{11\pi}{13}=\frac{1}{2}$.

64 Calculate the following sums:

a) $\cos a-\cos(a+h)+\cos(a+2h)-\ldots$
$\ldots+(-1)^{n-1}\cos[a+(n-1)h]$;

b) $\sin a-\sin(a+h)+\sin(a+2h)-\ldots$
$\ldots+(-1)^{n-1}\sin[a+(n-1)h]$.

65 If x has absolute value less than 1, show that the following series

a) $\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \dots + x^n\cos(\alpha+n\beta) + \dots,$

b) $\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \dots + x^n\sin(\alpha+n\beta) + \dots$

respectively converge to the following values:

$$\frac{\cos\alpha - x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}, \quad \frac{\sin\alpha - x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2}.$$

66 Calculate the following sums:

a) $\cos x + C_1^n\cos 2x + \dots + C_n^n\cos(n+1)x;$

b) $\sin x + C_1^n\sin 2x + \dots + C_n^n\sin(n+1)x.$

67 Calculate the following sums:

a) $\cos x - C_1^n\cos 2x + C_2^n\cos 3x - \dots + (-1)^n C_n^n\cos(n+1)x;$

b) $\sin x - C_1^n\sin 2x + C_2^n\sin 3x - \dots + (-1)^n C_n^n\sin(n+1)x.$

*68 Let the vectors defining the points 1, i on the Argand diagram be denoted respectively by $\overrightarrow{OA_1}$ and $\overrightarrow{OB}$ respectively. The perpendicular from 0 meets A_1B in A_2. The perpendicular from A_2 meets OA_1 in A_3, the perpendicular from A_3 meets A_1A_2 in A_4 and so on. In general, A_nA_{n+1} is perpendicular to $A_{n-2}A_{n-1}$. Calculate the limit of the sum $\overrightarrow{OA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots$

*69 Compute the following sum

$$\sin^2 x + \sin^2 3x + \ldots + \sin^2 (2n - 1) x.$$

70 Establish the following identities:

a) $\cos^2 x + \cos^2 2x + \ldots + \cos^2 nx = \frac{n}{2} + \frac{\cos (n+1) x \sin nx}{2 \sin x}$;

b) $\sin^2 x + \sin^2 2x + \ldots + \sin^2 nx = \frac{n}{2} - \frac{\cos (n+1) x \sin nx}{2 \sin x}$.

*71 Compute the sums:

a) $\cos^3 x + \cos^3 2x + \ldots + \cos^3 nx$;
b) $\sin^3 x + \sin^3 2x + \ldots + \sin^3 nx$.

*72 Compute the sums:

a) $\cos x + 2 \cos 2x + 3 \cos 3x + \ldots + n \cos nx$;
b) $\sin x + 2 \sin 2x + 3 \sin 3x + \ldots + n \sin nx$.

73 Let $\alpha = a + bi$. Compute the limit

$$\lim_{n \to \infty} \left(1 + \frac{\alpha}{n}\right)^n$$

74 Definition: $e^\alpha = \lim_{n \to \infty} \left(1 + \frac{\alpha}{n}\right)^n$.

Establish the following:

a) $e^{2\pi i} = 1$; b) $e^{\pi i} = -1$;
c) $e^{\alpha+\beta} = e^\alpha \cdot e^\beta$; d) $\left(e^\alpha\right)^k = e^{\alpha k}$

where k is an integer.

3. EQUATIONS OF THE THIRD AND FOURTH DEGREE

75 Use Cardan's formula to solve the following equations:

a) $x^3 - 6x + 9 = 0$; b) $x^3 + 12x + 63 = 0$;
c) $x^3 + 9x^2 + 18x + 28 = 0$; d) $x^3 + 6x^2 + 30x + 25 = 0$;
e) $x^3 - 6x + 4 = 0$; f) $x^3 + 6x + 2 = 0$;
g) $x^3 + 18x + 15 = 0$; h) $x^3 - 3x^2 - 3x + 11 = 0$;
i) $x^3 + 3x^2 - 6x + 4 = 0$; j) $x^3 + 9x - 26 = 0$;
k) $x^3 + 24x - 56 = 0$; l) $x^3 + 45x - 98 = 0$;
m) $x^3 + 3x^2 - 3x - 1 = 0$; n) $x^3 - 6x^2 + 57x - 196 = 0$;
o) $x^3 + 3x - 2i = 0$; p) $x^3 - 6ix + 4(1 - i) = 0$;
q) $x^3 - 3abx + a^3 + b^3 = 0$;
r) $x^3 - 3abfgx + f^2ga^3 + fg^2b^3 = 0$;
s) $x^3 - 4x - 1 = 0$; t) $x^3 - 4x + 2 = 0$.

*76 Use Cardan's formula to establish the following relation, where x_1, x_2, x_3 are the roots of the equation $x^3 + px + q = 0$.

$$(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 = -4p^3 - 27q^2,$$

(The expression $-4p^3 - 27q^2$ is called the discriminant of the equation $x^3 + px + q = 0$.)

*77 Solve the following equation:

$$(x^3 - 3qx + p^3 - 3pq)^2 - 4(px + q)^3 = 0.$$

*78 Find a formula for solving the equation

$$x^5 - 5ax^3 + 5a^2x - 2b = 0.$$

79 Solve each of the following.

a) $x^4 - 2x^3 + 2x^2 + 4x - 8 = 0$; b) $x^4 + 2x^3 - 2x^2 + 6x - 15 = 0$;
c) $x^4 - x^3 - x^2 + 2x - 2 = 0$; d) $x^4 - 4x^3 + 3x^2 + 2x - 1 = 0$;
e) $x^4 - 3x^3 + x^2 + 4x - 6 = 0$; f) $x^4 - 6x^3 + 6x^2 + 27x - 56 = 0$;
g) $x^4 - 2x^3 + 4x^2 - 2x + 3 = 0$; h) $x^4 - x^3 - 3x^2 + 5x - 10 = 0$;
i) $x^4 + 2x^3 + 8x^2 + 2x + 7 = 0$; j) $x^4 + 6x^3 + 6x^2 - 8 = 0$;
k) $x^4 - 6x^3 + 10x^2 - 2x - 3 = 0$; l) $x^4 - 2x^3 + 4x^2 + 2x - 5 = 0$;
m) $x^4 - x^3 - 3x^2 + x + 1 = 0$; n) $x^4 - x^3 - 4x^2 + 4x + 1 = 0$;
o) $x^4 - 2x^3 + x^2 + 2x - 1 = 0$; p) $x^4 - 4x^3 - 20x^2 - 8x + 4 = 0$;
q) $x^4 - 2x^3 + 3x^2 - 2x - 2 = 0$; r) $x^4 - x^3 + 2x - 1 = 0$;
s) $4x^4 - 4x^3 + 3x^2 - 2x + 1 = 0$; t) $4x^4 - 4x^3 - 6x^2 + 2x + 1 = 0$.

80 Ferrari's algorithm for solving the fourth degree equation $x^4 + ax^3 + bx^2 + cx + d = 0$ consists in writing the left member in the form

$$\left(x^2 + \frac{a}{2}x + \frac{\lambda}{2}\right)^2 - \left[\left(\frac{a^2}{4} + \lambda - b\right)x^2 + \left(\frac{a\lambda}{2} - c\right)x + \left(\frac{\lambda^2}{4} - d\right)\right]$$

and noting that if λ is properly chosen the square bracket will be a perfect square and thus the expression can be factored. A necessary condition that this occur is

$$\left(\frac{a\lambda}{2} - c\right)^2 - 4\left(\frac{a^2}{4} + \lambda - b\right)\left(\frac{\lambda^2}{4} - d\right) = 0,$$

Thus λ must be the solution of a "resolvent"

equation of the third degree. Once λ is determined the left member of the original equation can be factored.

Show that conversely the value of λ can be expressed in terms of the roots of the original fourth degree equation.

4. ROOTS OF UNITY

81 Find the roots of unity of the following orders:

a) 2; b) 3; c) 4; d) 6; 3) 8; f) 12; g) 24.

82 Find primitive roots of unity of the following orders:

a) 2; b) 3; c) 4; d) 6; e) 8; f) 12; g) 24.

83 To what exponent do the following quantities belong:

a) $z_k = \cos\frac{2k\pi}{180} + i\sin\frac{2k\pi}{180}$ for k = 27, 99, 137;

b) $z_k = \cos\frac{2k\pi}{144} + i\sin\frac{2k\pi}{144}$ for k = 10, 35, 60?

84 Find those 28-th units of unity that belong to exponent 7.

85 Find the exponent to which the various roots of unity belong, the degrees of the roots being
a) 16; b) 20; c) 24.

86 Find the cyclotomic polynomial $X_n(x)$ for the following values of n:

a) 1; b) 2; c) 3; d) 4; e) 5; f) 6;
g) 7; h) 8; i) 9; j) 10; k) 11; l) 12;
m) 15; n) 105.

*87 Let ϵ be a primitive 2n-th root of 1. Calculate the following sums:

$$1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1}\ .$$

*88 Calculate the sum of all n-th roots of unity.

*89 Calculate the sum of the k-th powers of all n-th roots of unity.

90 Calculate the sum of the m quantities $(x + a)^m$ where a runs through the m-th roots of unity.

*91 Let ϵ be an n-th root of unity. Calculate

$$1 + 2\varepsilon + 3\varepsilon^2 + \ldots + n\varepsilon^{n-1}$$

*92 Let ϵ be an n-th root of unity. Calculate

$$1 + 4\varepsilon + 9\varepsilon^2 + \ldots + n^2\varepsilon^{n-1}$$

93 Calculate the following sums:

a) $\cos\frac{2\pi}{n} + 2\cos\frac{4\pi}{n} + \ldots + (n-1)\cos\frac{2(n-1)\pi}{n}$;

b) $\sin\frac{2\pi}{n} + 2\sin\frac{4\pi}{n} + \ldots + (n-1)\sin\frac{2(n-1)\pi}{n}$.

*94 Compute the sum of the primitive roots of unity of respective degrees: a) 15; b) 24; c) 30.

95 Calculate the 5-th roots of unity, that is the solutions of the equation $x^5 - 1 = 0$, in both algebraic and polar form.

96 Use the result of problem 95 to compute $\sin 18^\circ$, $\cos 18^\circ$.

*97 Set up an algebraic equation the roots of which are the length of the sides of a regular 14-gon with circumradius 1.

*98 Factor the polynomial $x^n - 1$ into polynomials of first and second degree with real coefficients.

*99 Use the results of problem 98 to establish the following formulas:

a) $\sin\frac{\pi}{2m}\cdot\sin\frac{2\pi}{2m}\ldots\sin\frac{(m-1)\pi}{2m}=\frac{\sqrt{m}}{2^{m-1}}$;

b) $\sin\frac{\pi}{2m+1}\cdot\sin\frac{2\pi}{2m+1}\ldots\sin\frac{m\pi}{2m+1}=\frac{\sqrt{2m+1}}{2^m}$.

*100 Establish the following $\prod_{k=0}^{n-1}(a+b\varepsilon_k)=a^n+(-1)^{n-1}b^n$,

where

$$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}.$$

*101 Establish the following

$$\prod_{k=0}^{n-1}(\varepsilon_k^2-2\varepsilon_k\cos\theta+1)=2(1-\cos n\theta),$$

where $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

102 Establish the following

$$\prod_{k=0}^{n-1} \frac{(t+\varepsilon_k)^n - 1}{t} = \prod_{k=1}^{n-1} [t^n - (\varepsilon_k - 1)^n],$$

where $\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

*103 Find all complex numbers that satisfy the relation $\bar{x} = x^{n-1}$, where $\bar{x}$ is the complex conjugate of x.

104 Show that the roots of the equation $\lambda(z - a)^n + \mu(z - b)^n = 0$, where λ, μ, a, b are complex, lie on a circle which can in special cases reduce to a line segment (n is a rational integer).

*105 Solve each of the following:

a) $(x+1)^m - (x-1)^m = 0$; b) $(x+i)^m - (x-i)^m = 0$;
c) $x^n - nax^{n-1} - C_2^n a^2 x^{n-2} - \ldots - a^n = 0$.

106 Show that if A is a complex number of modulus 1, all the roots of the equation

$$\left(\frac{1+ix}{1-ix}\right)^m = A$$

are real and distinct.

*107 Solve the equation

$$\cos\varphi + C_1^n \cos(\varphi+\alpha)\,x + C_2^n \cos(\varphi+2\alpha)\,x^2 + \ldots$$
$$\ldots + C_n^n \cos(\varphi+n\alpha)\,x^n = 0.$$

Prove the following theorems:

108 The product of an a-th root of unity by a b-th root of unity is an ab-th root of unity.

109 Let a, b be relatively prime integers. Then $x^a - 1$, $x^b - 1$ have only the single trivial common root.

110 Let a, b be relatively prime integers. Then all the ab-th roots of unity can be obtained as products of a-th roots of unity and b-th roots of unity.

111 Let a, b be relatively prime. Then the product of a primitive a-th root of unity by a primitive b-th root of unity is a primitive ab-th root of unity, and conversely.

112 Let $\varphi(n)$ be the number of primitive n-th roots of unity. Show that if a, b are relatively prime the relation $\varphi(ab) = \varphi(a)\,\varphi(b)$ holds.

*113 Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers (canonical decomposition of n). Show that

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right).$$

114 Show that the number of primitive n-th roots of unity, n > 2, is even.

115 If p is a prime number, calculate the polynomial $X_p(x)$, the cyclotomic polynomial.

*116 If p is a prime number, calculate the cyclotomic polynomial $X_{p^m}(x)$.

*117 If n is an odd number, $n > 1$, show that $X_{2n}(x) = X_n(-x)$.

118 If d is a product of prime numbers each of which divides n, then every primitive nd-th root of 1 is a d-th root of a primitive n-th root of 1, and conversely.

*119 Let the canonical decomposition of n be $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers. Then we have
$X_n(x) = X_{n'}(x^{n''})$, where

$$n' = p_1 p_2 \ldots p_k; \quad n'' = \frac{n}{n'}.$$

*120 Let $\mu(n)$ be the sum of the primitive n-th roots of 1. Show that $\mu(n)$ is 0 whenever n is divisible by the square of a prime number; $\mu(n)$ is 1 if n is the product of an even number of distinct prime numbers; $\mu(n) = -1$ if n is the product of an odd number of distinct prime numbers.

121 Let d run through all divisors of n, $n \neq 1$. Show that $\sum \mu(d) = 0$.

*122 Establish the relation

$$X_n(x) = \Pi (x^d - 1)^{\mu\left(\frac{n}{d}\right)}$$

where d runs through the divisors of n.

*123 Calculate $X_n(1)$.

*124 Calculate $X_n(-1)$.

*125 Find the value of the sum of products of all possible pairs of primitive n-th roots of 1.

*126 Find $|S|$ where $S = 1 + \varepsilon + \varepsilon^4 + \varepsilon^9 + \ldots + \varepsilon^{(n-1)^2}$ and where ϵ is a primitive n-th root of 1.

CHAPTER II - PROBLEMS

COMPUTATION OF DETERMINANTS

1. SECOND AND THIRD ORDER DETERMINANTS

Calculate the determinants of the following matrices:

127

a) $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$; b) $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$; c) $\begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$;

d) $\begin{bmatrix} a & c+di \\ c-di & b \end{bmatrix}$; e) $\begin{bmatrix} \alpha+\beta i & \gamma+\delta i \\ \gamma-\delta i & \alpha-\beta i \end{bmatrix}$; f) $\begin{bmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{bmatrix}$;

g) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{bmatrix}$; h) $\begin{bmatrix} \operatorname{tg}\alpha & -1 \\ 1 & \operatorname{tg}\alpha \end{bmatrix}$; i) $\begin{bmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{bmatrix}$;

j) $\begin{bmatrix} 1 & \lg_b a \\ \lg_a b & 1 \end{bmatrix}$; k) $\begin{bmatrix} a+b & b+d \\ a+c & c+d \end{bmatrix}$; l) $\begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$;

m) $\begin{bmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{bmatrix}$; n) $\begin{bmatrix} \omega & \omega \\ -1 & \omega \end{bmatrix}$,

where

$$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3};$$

o) $\begin{bmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{bmatrix}$,

where

$$\varepsilon = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}.$$

128

a) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$; b) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$; c) $\begin{bmatrix} a & a & a \\ -a & a & x \\ -a & -a & x \end{bmatrix}$;

d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$; e) $\begin{bmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{bmatrix}$;

f) $$\begin{bmatrix} 1 & \cos\frac{\pi}{3}+i\sin\frac{\pi}{3} & \cos\frac{\pi}{4}+i\sin\frac{\pi}{4} \\ \cos\frac{\pi}{3}-i\sin\frac{\pi}{3} & 1 & \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} \\ \cos\frac{\pi}{4}-i\sin\frac{\pi}{4} & \cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3} & 1 \end{bmatrix};$$

g) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \text{ where } \omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3};$$

h) $$\begin{bmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{bmatrix}, \text{ where } \omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}.$$

2. PERMUTATIONS

129 Write out the transpositions whose product changes the permutation 1, 2, 4, 3, 5 into the permutation 2, 5, 3, 4, 1.

130 How many inversions from natural order 1, 2, 3, 4, 5, 6, 7, 8, 9 are there in the following sequences:

a) 1, 3, 4, 7, 8, 2, 6, 9, 5;

b) 2, 1, 7, 9, 8, 6, 3, 5, 4;

c) 9, 8, 7, 6, 5, 4, 3, 2, 1.

131 The permutations below are rearrangements of the natural order 1, 2, 3, 4, 5, 6, 7, 8, 9. Determine i and k so that

a) the permutation 1, 2, 7, 4, i, 5, 6, k, 9 is an even permutation;

b) the permutation 1, i, 2, 5, k, 4, 8, 9, 7 is an odd permutation.

*132 How many inversions from the natural order $1, 2, \cdots, n$ are there in the permutation $n, n-1, \cdots, 2, 1$.

*133 Suppose the permutation $\alpha_1, \alpha_2, \cdots, \alpha_n$ has I inversions. How many inversions are there in the permutation $\alpha_n, \alpha_{n-1}, \cdots, \alpha_2, \alpha_1$?

134 How many permutations from the natural order $1, 2, \cdots, 2n$ are there in the permutations

a) $1, 3, 5, 7, \cdots, 2n-1, 2, 4, 6, \cdots, 2n$;

b) $2, 4, 6, 8, \cdots, 2n, 1, 3, 5, \cdots, 2n-1$.

135 How many inversions from the natural order $1, 2, 3, \cdots, 3n$ are there in the permutations

a) $3, 6, 9, \cdots, 3n, 1, 4, 7, \cdots, 3n-2, 2, 5, \cdots, 3n-1$;

b) $1, 4, 7, \cdots, 3n-2, 2, 5, \cdots, 3n-1, 3, 6, \cdots, 3n$?

136 If we think of $1, 2, \cdots, n$ as the original order and suppose that the order $a_1, a_2, \cdots, a_n$ has I inversions, show that by changing the roles of these two permutations the number of inversions in going from the second order as the natural order to the first as an inverted order is the same.

137 Find whether the permutation t, h, r, m, i, a, g, o, l, is an even or odd permutation, assuming that the natural order of the letters is that in the word:

a) logarithm;

b) algorithm.

3. DEFINITION OF DETERMINANTS

138 If a determinant is defined as the sum of n! terms, in the usual way, what sign must prefix each of the following terms in the expansion of the determinant of the 6-th order matrix $[a_{ij}]$:

a) $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$; b) $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$?

139 Which of the following products actually appears in the expansion of the determinant of the 5-th order matrix $[a_{ij}]$:

a) $a_{13}a_{24}a_{23}a_{41}a_{55}$; b) $a_{21}a_{13}a_{34}a_{55}a_{42}$?

140 In the expansion of the determinant of the 5-th order matrix $[a_{ij}]$ $a_{1i}a_{32}a_{4k}a_{25}a_{53}$ appears with a plus sign. Find i and k.

141 Write down all the terms that appear prefixed by minus sign, in the expansion of the determinant of the 4-th order matrix $[a_{ij}]$ and which contain the factor a_{23}.

142 Which terms of the form $a_{14}a_{23}a_{3\alpha_3}a_{4\alpha_4}a_{5\alpha_5}$ appear in the expansion of the determinant of the 5-th order matrix $[a_{ij}]$? The sum of these terms naturally has the common factor $a_{14}a_{23}$. What is a convenient way of writing the other factor?

143 One of the terms in the expansion of the determinant of the n-th order matrix $[a_{ij}]$ is the product of the elements in the principal diagonal. What sign is prefixed to this term in the expansion of the determinant?

144 One of the terms in the expansion of the determinant of the n-th order matrix $[a_{ij}]$ is the product of the elements in the secondary diagonal. What sign is prefixed to this term in the expansion of the determinant?

145 Use only the definition of the determinant of the matrix to show that the determinant of the following matrix is 0:

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ a_1 & a_2 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 & 0 \end{bmatrix}$$

146 Use the definition of the determinant of the matrix to calculate the coefficients of x^4 and x^3 in the

expansion of the determinant:

$$f(x) = \det \begin{bmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}.$$

147 Evaluate the following determinants:

a) $\det \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & n \end{bmatrix}$; b) $\det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$;

c) $\det \begin{bmatrix} 1 & a & a & \dots & a \\ 0 & 2 & a & \dots & a \\ 0 & 0 & 3 & \dots & a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & n \end{bmatrix}$.

Note: In all succeeding exercises assume that the order of the matrix is n unless otherwise given.

148 Define $F(x) = x(x - 1)(x - 2) \cdots (x - n + 1)$. Calculate the values:

a) $\det \begin{bmatrix} F(0) & F(1) & F(2) & \dots & F(n) \\ F(1) & F(2) & F(3) & \dots & F(n+1) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F(n) & F(n+1) & F(n+2) & \dots & F(2n) \end{bmatrix}$;

b) $\det \begin{bmatrix} F(a) & F'(a) & F''(a) & \dots & F^{(n)}(a) \\ F'(a) & F''(a) & F'''(a) & \dots & F^{(n+1)}(a) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F^{(n)}(a) & F^{(n+1)}(a) & F^{(n+2)}(a) & \dots & F^{(2n)}(a) \end{bmatrix}.$

4. FUNDAMENTAL PROPERTIES OF DETERMINANTS

*149 Let an n-th order matrix be such that the ik element a_{ik} is the complex conjugate of ki element a_{ki}. Show that the determinant of the matrix is real.

*150 Let the matrix $[a_{ij}]$ have odd order, and suppose the sum of the symmetrically situated elements is 0:

$$a_{ik} + a_{ki} = 0.$$

Show that the determinant of the matrix is 0 (th determinant of a skew-symmetric matrix of odd order is 0).

151 Suppose the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

is Δ. Find the value of the determinant of the matrix

$$\begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \\ a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$

152 How is the value of the determinant of a matrix changed if the columns are written in inverse order?

*153 Calculate the value of the sum of determinants

$$\sum \det \begin{bmatrix} a_{1\alpha_1} & a_{1\alpha_2} & \dots & a_{1\alpha_n} \\ a_{2\alpha_1} & a_{2\alpha_2} & \dots & a_{2\alpha_n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n\alpha_1} & a_{n\alpha_2} & \dots & a_{n\alpha_n} \end{bmatrix},$$

the sum being extended over all possible permutations α_1 $\alpha_2, \cdots, \alpha_n$.

*154 Let a_1, $a_2, \cdots$, a_{n-1} be different numbers. Solve the following equations:

a) $$\det \begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{n-1} \end{bmatrix} = 0,$$

b) $\det \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1-x & 1 & \dots & 1 \\ 1 & 1 & 2-x & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & & (n-1)-x \end{bmatrix} = 0;$

c) $\det \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_1+a_2-x & a_3 & \dots & a_n \\ a_1 & a_2 & a_2+a_3-x & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \dots & a_{n-1}+a_n-x \end{bmatrix} = 0.$

*155 Noting that each of the numbers 204, 527, and 255 is divisible by 17, show that the determinant of the matrix

$$\begin{bmatrix} 2 & 0 & 4 \\ 5 & 2 & 7 \\ 2 & 5 & 5 \end{bmatrix}$$

is divisible by 17.

*156 Calculate the determinant of the matrix

$$\begin{bmatrix} \alpha^2 & (\alpha+1)^2 & (\alpha+2)^2 & (\alpha+3)^2 \\ \beta^2 & (\beta+1)^2 & (\beta+2)^2 & (\beta+3)^2 \\ \gamma^2 & (\gamma+1)^2 & (\gamma+2)^2 & (\gamma+3)^2 \\ \delta^2 & (\delta+1)^2 & (\delta+2)^2 & (\delta+3)^2 \end{bmatrix}.$$

157 Establish the following relation between the two determinants shown

$$\det \begin{bmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{bmatrix} = 2 \det \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}.$$

158 Evaluate the determinant of the following matrix by writing it as a sum of determinants

$$\begin{bmatrix} am+bp & an+bq \\ cm+dp & cn+dq \end{bmatrix}$$

159 In the case of each matrix, calculate the sum of all the algebraic co-factors of the individual elements in the expansion of the corresponding determinant:

$$\text{a)} \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}; \quad \text{b)} \begin{bmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_2 & 0 \\ . & . & . & . & . \\ a_n & 0 & \dots & 0 & 0 \end{bmatrix}.$$

160 Expand the determinant of the following matrix by co-factors of the elements of the third row and carry out the computations

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & c & d \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

161 Expand the determinant of the following matrix by co-factors of the elements of the last column and carry out the computations

$$\begin{bmatrix} 2 & 1 & 1 & x \\ 1 & 2 & 1 & y \\ 1 & 1 & 2 & z \\ 1 & 1 & 1 & t \end{bmatrix}.$$

162 Expand the determinant of the following matrix by co-factors of the elements of the first column and carry out the calculations

$$\begin{bmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{bmatrix}.$$

5. EVALUATION OF DETERMINANTS

Calculate the values of determinants of the following matrices:

***163** $\begin{bmatrix} 13547 & 13647 \\ 28423 & 28523 \end{bmatrix}$. **164** $\begin{bmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{bmatrix}$.

165 $\begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. **166** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$. **167** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

168 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}$. **169** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{bmatrix}$.

170 $\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}$. **171** $\begin{bmatrix} 5 & 6 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$.

172 $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{bmatrix}$. **173** $\begin{bmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{bmatrix}$.

174 $\begin{bmatrix} x & 0 & -1 & 1 & 0 \\ 1 & x & -1 & 1 & 0 \\ 1 & 0 & x-1 & 0 & 1 \\ 0 & 1 & -1 & x & 1 \\ 0 & 1 & -1 & 0 & x \end{bmatrix}$. **175** $\begin{bmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{bmatrix}$.

176
$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{bmatrix}.$$

177
$$\begin{bmatrix} \cos(a-b) & \cos(b-c) & \cos(c-a) \\ \cos(a+b) & \cos(b+c) & \cos(c+a) \\ \sin(a+b) & \sin(b+c) & \sin(c+a) \end{bmatrix}.$$

178 $\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$. ***179** $\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -1 & -2 & -3 & \dots & 0 \end{bmatrix}$.

***180**

$$\begin{bmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1+b_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2+b_2 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_1 & a_2 & \dots & a_n+b_n \end{bmatrix}.$$

***181**

$$\begin{bmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} & x_n \\ 1 & x & x_2 & \dots & x_{n-1} & x_n \\ 1 & x_1 & x & \dots & x_{n-1} & x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_1 & x_2 & \dots & x & x_n \\ 1 & x_1 & x_2 & \dots & x_{n-1} & x \end{bmatrix}.$$

***182**

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 3 & 3 & \dots & n-1 & n \\ 1 & 2 & 5 & \dots & n-1 & n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & \dots & 2n-3 & n \\ 1 & 2 & 3 & \dots & n-1 & 2n-1 \end{bmatrix}.$$

***183**

$$\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 2 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 2 & 2 & \dots & n \end{bmatrix}.$$

***184**

$$\begin{bmatrix} 1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1-b_1 & b_2 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1-b_2 & b_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1-b_{n-1} & b_n \\ 0 & 0 & 0 & 0 & \dots & -1 & 1-b_n \end{bmatrix}.$$

***185**

$$\begin{bmatrix} a & a+h & a+2h & \dots & a+(n-1)h \\ -a & a & 0 & \dots & 0 \\ 0 & -a & a & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a \end{bmatrix}.$$

***186**

$$\begin{bmatrix} a & -(a+h) & \dots & (-1)^{n-3}[a+(n-2)h] & (-1)^{n-1}[a+(n-1)h] \\ a & a & \dots & 0 & 0 \\ 0 & a & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & a & a \end{bmatrix}.$$

***187**
$$\begin{bmatrix} 1 & C_1^n & C_2^n & C_3^n & \dots & C_{n-2}^n & C_{n-1}^n & C_n^n \\ 1 & C_1^{n-1} & C_2^{n-1} & C_3^{n-1} & \dots & C_{n-2}^{n-1} & C_{n-1}^{n-1} & 0 \\ 1 & C_1^{n-2} & C_2^{n-2} & C_3^{n-2} & \dots & C_{n-2}^{n-2} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & C_1^2 & C_2^2 & 0 & \dots & 0 & 0 & 0 \\ 1 & C_1^1 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_n \end{bmatrix}.$$

***188**
$$\begin{bmatrix} a_0 & -1 & 0 & \dots & 0 & 0 \\ a_1 & x & -1 & \dots & 0 & 0 \\ a_2 & 0 & x & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & 0 & 0 & \dots & x & -1 \\ a_n & 0 & 0 & \dots & 0 & x \end{bmatrix}.$$

***189**
$$\begin{bmatrix} n & n-1 & n-2 & \dots & 3 & 2 & 1 \\ -1 & x & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & x & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -1 & x & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & x \end{bmatrix}.$$

*190 Let $f(x)$ be defined as the determinant of the following matrix

$$f(x) = \det\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & x \\ 1 & 2 & 0 & 0 & \dots & 0 & x^2 \\ 1 & 3 & 3 & 0 & \dots & 0 & x^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & n & C_2^n & C_3^n & \dots & C_{n-1}^n & x^n \\ 1 & n+1 & C_2^{n+1} & C_3^{n+1} & \dots & C_{n-1}^{n+1} & x^{n+1} \end{bmatrix}.$$

Calculate the value $f(x + 1) - f(x)$.

Calculate the determinants of the following matrices:

***191** $$\begin{bmatrix} x & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & x & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & x & \dots & a_{n-1} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \dots & x & 1 \\ a_1 & a_2 & a_3 & \dots & a_n & 1 \end{bmatrix}.$$

***192** $$\begin{bmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a & a & a & \dots & x \end{bmatrix}.$$

193 $$\begin{bmatrix} x & a & a & \dots & a & a \\ -a & x & a & \dots & a & a \\ -a & -a & x & \dots & a & a \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a & -a & -a & \dots & -a & x \end{bmatrix}.$$

***194** $$\begin{bmatrix} -a_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & -a_2 & a_2 & \dots & 0 & 0 \\ 0 & 0 & -a_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -a_n & a_n \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

***195** $$\begin{bmatrix} a_1 & -a_2 & 0 & \dots & 0 & 0 \\ 0 & a_2 & -a_3 & \dots & 0 & 0 \\ 0 & 0 & a_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_{n-1} & -a_n \\ 1 & 1 & 1 & \dots & 1 & 1+a_n \end{bmatrix}.$$

***196** $$\begin{bmatrix} h & -1 & 0 & 0 & \dots & 0 \\ hx & h & -1 & 0 & \dots & 0 \\ hx^2 & hx & h & -1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ hx^n & hx^{n-1} & hx^{n-2} & hx^{n-3} & \dots & h \end{bmatrix}.$$

***197** $$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & x & \dots & x & x \\ 1 & x & 0 & \dots & x & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x & x & \dots & 0 & x \\ 1 & x & x & \dots & x & 0 \end{bmatrix}.$$

***198** $$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_1+a_2 & \dots & a_1+a_n \\ 1 & a_2+a_1 & 0 & \dots & a_2+a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_n+a_1 & a_n+a_2 & \dots & 0 \end{bmatrix}.$$

***199** $$\begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 1 & 1 & \dots & 1 & 1-n \\ 1 & 1 & 1 & \dots & 1-n & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1-n & 1 & \dots & 1 & 1 \end{bmatrix}.$$

*200 (This matrix has order n + 1.)

$$\begin{bmatrix} 2 & 1-\frac{1}{n} & 1-\frac{1}{n} & \dots & 1-\frac{1}{n} \\ 1-\frac{1}{n} & 2 & 1-\frac{1}{n} & \dots & 1-\frac{1}{n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1-\frac{1}{n} & 1-\frac{1}{n} & 1-\frac{1}{n} & \dots & 2 \end{bmatrix}$$

***201**

$$\begin{bmatrix} 1 & a & a^2 & a^3 & \dots & a^n \\ x_{11} & 1 & a & a^2 & \dots & a^{n-1} \\ x_{21} & x_{22} & 1 & a & \dots & a^{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & \dots & 1 \end{bmatrix}.$$

***202**

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 1 & 2 & 3 & \dots & n-1 \\ 3 & 2 & 1 & 2 & \dots & n-2 \\ 4 & 3 & 2 & 1 & \dots & n-3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n & n-1 & n-2 & n-3 & \dots & 1 \end{bmatrix}.$$

***203**

$$\begin{bmatrix} a_0 & b_1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & -b_0 & b_2 & 0 & \dots & 0 & 0 \\ a_2 & 0 & -b_1 & b_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & 0 & 0 & 0 & \dots & -b_{n-2} & b_n \\ a_n & 0 & 0 & 0 & \dots & 0 & -b_{n-1} \end{bmatrix}.$$

***204**

$$\begin{bmatrix} a & a^2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2a+b & (a+b)^2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2a+3b & (a+2b)^2 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 2a+(2n-1)b & (a+nb)^2 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2a+(2n+1)b \end{bmatrix}.$$

***205**

$$\begin{bmatrix} x & y & 0 & \dots & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & x & y \\ y & 0 & 0 & \dots & 0 & x \end{bmatrix}.$$

***206**

$$\begin{bmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \cdot & \cdot & \cdot & \cdot \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{bmatrix}.$$

207

$$\begin{bmatrix} a_1-b_1 & a_1-b_2 & \dots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \dots & a_2-b_n \\ \cdot & \cdot & \cdot & \cdot \\ a_n-b_1 & a_n-b_2 & \dots & a_n-b_n \end{bmatrix}.$$

***208**

$$\begin{bmatrix} 1+a_1+x_1 & a_1+x_2 & \dots & a_1+x_n \\ a_2+x_1 & 1+a_2+x_2 & \dots & a_2+x_n \\ \cdot & \cdot & \cdot & \cdot \\ a_n+x_1 & a_n+x_2 & \dots & 1+a_n+x_n \end{bmatrix}.$$

209

$$\begin{bmatrix} a^n-\alpha & a^{n+1}-\alpha & \dots & a^{n+p-1}-\alpha \\ a^{n+p}-\alpha & a^{n+p+1}-\alpha & \dots & a^{n+2p-1}-\alpha \\ \cdot & \cdot & \cdot & \cdot \\ a^{n+p(p-1)}-\alpha & a^{n+p(p-1)+1}-\alpha & \dots & a^{n+p^2-1}-\alpha \end{bmatrix}.$$

210 Let $f_1(x), f_2(x), \cdots, f_n(x)$ be polynomials in x of degree not exceeding $n-2$; let the numbers $a_1, a_2, \cdots, a_n$ be arbitrary. Show that the determinant of the following matrix is 0.

$$\begin{bmatrix} f_1(a_1) & f_1(a_2) & \dots & f_1(a_n) \\ f_2(a_1) & f_2(a_2) & \dots & f_2(a_n) \\ \cdot & \cdot & \cdot & \cdot \\ f_n(a_1) & f_n(a_2) & \dots & f_n(a_n) \end{bmatrix}$$

Calculate the determinants of the following matrices:

***211**

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ -1 & x & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & x \end{bmatrix}.$$

***212**

$$\begin{bmatrix} a_1+x_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ -x_1 & x_2 & 0 & \dots & 0 & 0 \\ 0 & -x_2 & x_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -x_{n-1} & x_n \end{bmatrix}.$$

***213**

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -y_1 & x_1 & 0 & \dots & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -y_n & x_n \end{bmatrix}.$$

***214**

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & \dots & 0 \\ 1 & 0 & a_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \dots & a_n \end{bmatrix}.$$

***215**

$$\begin{bmatrix} n!\,a_0 & (n-1)!\,a_1 & (n-2)!\,a_2 & \dots & a_n \\ -n & x & 0 & \dots & 0 \\ 0 & -(n-1) & x & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & x \end{bmatrix}.$$

216

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & a_1 & 0 & 0 & 0 \\ 1 & 1 & a_2 & 0 & 0 \\ 1 & 0 & 1 & a_3 & 0 \\ 1 & 0 & 0 & 1 & a_4 \end{bmatrix}.$$

Generalize problem 216 to the case of an arbitrary n-th order matrix with the same pattern; namely 1 is in the first column, 1 in the upper right-hand corner, and 1 in the first diagonal.

Calculate the determinants of the following matrices:

217

$$\begin{bmatrix} \alpha+\beta & \alpha\beta & 0 & \dots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{bmatrix}.$$

218

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 2 \end{bmatrix}.$$

***219**

$$\begin{bmatrix} 2\cos\theta & 1 & 0 & \dots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 2\cos\theta \end{bmatrix}.$$

220

$$\begin{bmatrix} \cos\theta & 1 & 0 & 0 \\ 1 & 2\cos\theta & 1 & 0 \\ 0 & 1 & 2\cos\theta & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 2\cos\theta \end{bmatrix}.$$

***221**

$$\begin{bmatrix} x & 1 & 0 & \dots & 0 \\ 1 & x & 1 & \dots & 0 \\ 0 & 1 & x & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & x \end{bmatrix}.$$

***222**

$$\begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \dots & x_1y_n \\ x_1y_2 & x_2y_2 & x_2y_3 & \dots & x_2y_n \\ x_1y_3 & x_2y_3 & x_3y_3 & \dots & x_3y_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1y_n & x_2y_n & x_3y_n & \dots & x_ny_n \end{bmatrix}.$$

***223**

$$\begin{bmatrix} 1+a_1 & 1 & 1 & \dots & 1 \\ 1 & 1+a_2 & 1 & \dots & 1 \\ 1 & 1 & 1+a_3 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \dots & 1+a_n \end{bmatrix}.$$

224

$$\begin{bmatrix} 1 & 1 & \dots & 1 & a_1+1 \\ 1 & 1 & \dots & a_2+1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n-1}+1 & \dots & 1 & 1 \\ a_n+1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

***225**

$$\begin{bmatrix} a_1 & x & x & \dots & x \\ x & a_2 & x & \dots & x \\ x & x & a_3 & \dots & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \dots & a_n \end{bmatrix}.$$

***226** $$\begin{bmatrix} x_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_1 & x_2 & a_3 & \dots & a_{n-1} & a_n \\ a_1 & a_2 & x_3 & \dots & a_{n-1} & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \dots & x_{n-1} & a_n \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & x_n \end{bmatrix}.$$

***227** $$\begin{bmatrix} x_1 & a_2b_1 & a_3b_1 & \dots & a_nb_1 \\ a_1b_2 & x_2 & a_3b_2 & \dots & a_nb_2 \\ a_1b_3 & a_2b_3 & x_3 & \dots & a_nb_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1b_n & a_2b_n & a_3b_n & \dots & x_n \end{bmatrix}.$$

***228** $$\begin{bmatrix} x_1 - m & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 - m & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 - m & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1 & x_2 & x_3 & \dots & x_n - m \end{bmatrix}.$$

229 Solve the following equation for x:

$$\det \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n - \alpha_n x \\ a_1 & a_2 & \dots & a_{n-1} - \alpha_{n-1}x & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 - \alpha_1 x & a_2 & \dots & a_{n-1} & a_n \end{bmatrix} = 0.$$

Calculate the determinants of the following matrices:

***230**

$$\begin{bmatrix} a & 0 & 0 & \dots & 0 & 0 & b \\ 0 & a & 0 & \dots & 0 & b & 0 \\ 0 & 0 & a & \dots & b & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & b & \dots & a & 0 & 0 \\ 0 & b & 0 & \dots & 0 & a & 0 \\ b & 0 & 0 & \dots & 0 & 0 & a \end{bmatrix}.$$

(Order 2n)

***231**

$$\begin{bmatrix} 1 & -b & -b & -b & \dots & -b \\ 1 & na & -2b & -3b & \dots & -(n-1)b \\ 1 & (n-1)a & a & -3b & \dots & -(n-1)b \\ 1 & (n-2)a & a & a & \dots & -(n-1)b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2a & a & a & \dots & a \end{bmatrix}.$$

***232**

$$\begin{bmatrix} (x-a_1)^2 & a_2^2 & \dots & a_n^2 \\ a_1^2 & (x-a_2)^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^2 & a_2^2 & \dots & (x-a_n)^2 \end{bmatrix}.$$

***233**

$$\begin{bmatrix} (x-a_1)^2 & a_1a_2 & \dots & a_1a_n \\ a_1a_2 & (x-a_2)^2 & \dots & a_2a_n \\ \dots & \dots & \dots & \dots \\ a_1a_n & a_2a_n & \dots & (x-a_n)^2 \end{bmatrix}.$$

***234**

$$\begin{bmatrix} 1-b_1 & b_2 & 0 & 0 & \dots & 0 \\ -1 & 1-b_2 & b_3 & 0 & \dots & 0 \\ 0 & -1 & 1-b_3 & b_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1-b_n \end{bmatrix}.$$

***235**

$$\begin{bmatrix} 0 & a_2 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ b_1 & 0 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ b_1 & b_2 & 0 & a_4 & \dots & a_{n-1} & a_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_1 & b_2 & b_3 & b_4 & \dots & 0 & a_n \\ b_1 & b_2 & b_3 & b_4 & \dots & b_{n-1} & 0 \end{bmatrix}.$$

***236**

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 2 & 3 & 4 & \dots & n-1 \\ 1 & x & 1 & 2 & 3 & \dots & n-2 \\ 1 & x & x & 1 & 2 & \dots & n-3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & x & x & \dots & 1 \end{bmatrix}.$$

***237**

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n \\ x & 1 & 2 & 3 & \dots & n-1 \\ x & x & 1 & 2 & \dots & n-2 \\ x & x & x & 1 & \dots & n-3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & x & x & x & \dots & 1 \end{bmatrix}.$$

***238**

$$\begin{bmatrix} a_0x^n & a_1x^{n-1} & a_2x^{n-2} & \dots & a_{n-1}x & a_n \\ a_0x & b_1 & 0 & \dots & 0 & 0 \\ a_0x^2 & a_1x & b_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0x^{n-1} & a_1x^{n-2} & a_2x^{n-3} & \dots & b_{n-1} & 0 \\ a_0x^n & a_1x^{n-1} & a_2x^{n-2} & \dots & a_{n-1}x & b_n \end{bmatrix}.$$

*239 Show

$$\det\begin{bmatrix} a_{00}x^n & a_{01}x^{n-1} & a_{02}x^{n-2} & \dots & a_{0n} \\ a_{10}x & a_{11} & 0 & \dots & 0 \\ a_{20}x^2 & a_{21}x & a_{22} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n0}x^n & a_{n1}x^{n-1} & a_{n2}x^{n-2} & \dots & a_{nn} \end{bmatrix} = x^n \cdot \det\begin{bmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} \\ a_{10} & a_{11} & 0 & \dots & 0 \\ a_{20} & a_{21} & a_{22} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Calculate the determinants of the following matrices:

***240**

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & C_1^2 & C_1^3 & \dots & C_1^n \\ 1 & C_2^3 & C_2^4 & \dots & C_2^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & C_{n-1}^n & C_{n-1}^{n+1} & \dots & C_{n-1}^{2n-2} \end{bmatrix}.$$

***241**

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ C_1^m & C_1^{m+1} & \dots & C_1^{m+n} \\ C_2^{m+1} & C_2^{m+2} & \dots & C_2^{m+n+1} \\ \cdot & \cdot & \cdot & \cdot \\ C_n^{m+n-1} & C_n^{m+n} & \dots & C_n^{m+2n-1} \end{bmatrix}.$$

***242**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & C_1^2 & C_2^2 & 0 & \dots & 0 \\ 1 & C_1^3 & C_2^3 & C_3^3 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & C_1^n & C_2^n & C_3^n & \dots & C_{n-1}^n \end{bmatrix}.$$

***243**

$$\begin{bmatrix} C_k^m & C_{k+1}^m & \dots & C_{k+n}^m \\ C_k^{m+1} & C_{k+1}^{m+1} & \dots & C_{k+n}^{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ C_k^{m+n} & C_{k+1}^{m+n} & \dots & C_{k+n}^{m+n} \end{bmatrix}.$$

***244**

$$\begin{bmatrix} C_m^{k+m} & C_m^{k+m+1} & \dots & C_m^{k+2m} \\ C_m^{k+m+1} & C_m^{k+m+2} & \dots & C_m^{k+2m+1} \\ \cdot & \cdot & \cdot & \cdot \\ C_m^{k+2m} & C_m^{k+2m+1} & \dots & C_m^{k+3m} \end{bmatrix}.$$

***245**

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & C_1^1 & 0 & \dots & 0 & x \\ 1 & C_1^2 & C_2^2 & \dots & 0 & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & C_1^n & C_2^n & \dots & C_{n-1}^n & x^n \end{bmatrix}.$$

***246** $$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 1 \\ 1 & 1! & 0 & 0 & \dots & x \\ 1 & 2 & 2! & 0 & \dots & x^2 \\ 1 & 3 & 3\cdot 2 & 3! & \dots & x^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & n & n(n-1) & n(n-1)(n-2) & \dots & x^n \end{bmatrix}.$$

***247**
$$\begin{bmatrix} \alpha & \alpha+\delta & \alpha+2\delta & \alpha+3\delta & \dots & \alpha+(n-1)\delta \\ \alpha & 2\alpha+\delta & 3\alpha+3\delta & 4\alpha+6\delta & \dots & C_1^n\alpha+C_2^n\delta \\ \alpha & 3\alpha+\delta & 6\alpha+4\delta & 10\alpha+10\delta & \dots & C_2^{n+1}\alpha+C_3^{n+1}\delta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha & C_{n-1}^n\alpha+\delta & C_{n-1}^{n+1}\alpha+C_n^{n+1}\delta & C_{n-1}^{n+2}\alpha+C_n^{n+2}\delta & \dots & C_{n-1}^{2n-2}\alpha+C_n^{2n-2}\delta \end{bmatrix}.$$

***248** $$\begin{bmatrix} x & y & y & \dots & y & y \\ z & x & y & \dots & y & y \\ z & z & x & \dots & y & y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ z & z & z & \dots & x & y \\ z & z & z & \dots & z & x \end{bmatrix}.$$

***249** $$\begin{bmatrix} a & a & a & \dots & a & 0 \\ a & a & a & \dots & 0 & b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a & 0 & b & \dots & b & b \\ 0 & b & b & \dots & b & b \end{bmatrix}.$$

250 $$\begin{bmatrix} a_1 & x & x & \dots & x \\ y & a_2 & x & \dots & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y & y & y & \dots & a_n \end{bmatrix}.$$

251 $$\begin{bmatrix} c_1 & a & a & \dots & a & 1 \\ b & c_2 & a & \dots & a & 1 \\ b & b & c_3 & \dots & a & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & b & \dots & c_n & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}.$$

***252** $$\begin{bmatrix} \lambda & a & a & a & \dots & a \\ b & \alpha & \beta & \beta & \dots & \beta \\ b & \beta & \alpha & \beta & \dots & \beta \\ b & \beta & \beta & \alpha & \dots & \beta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b & \beta & \beta & \beta & \dots & \alpha \end{bmatrix}.$$

***253** $$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & 1 \\ 3 & 4 & 5 & \dots & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ n & 1 & 2 & \dots & n-1 \end{bmatrix}.$$

***254** $$\begin{bmatrix} a & a+h & a+2h & \dots & a+(n-1)h \\ a+h & a+2h & a+3h & \dots & a \\ a+2h & a+3h & a+4h & \dots & a+h \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a+(n-1)h & a & a+h & \dots & a+(n-2)h \end{bmatrix}.$$

255 $\begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ x^{n-1} & 1 & x & \dots & x^{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x^2 & x^3 & \dots & 1 \end{vmatrix}$.

***256** $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$.

257 $\begin{vmatrix} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a \end{vmatrix}$.

***258** $\begin{vmatrix} x & a_1 & a_2 & \dots & a_n \\ a_1 & x & a_2 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \dots & x \end{vmatrix}$.

(order n + 1)

***259** $\begin{vmatrix} \cos^{n-1}\varphi_1 & \cos^{n-2}\varphi_1 & \dots & \cos\varphi_1 & 1 \\ \cos^{n-1}\varphi_2 & \cos^{n-2}\varphi_2 & \dots & \cos\varphi_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cos^{n-1}\varphi_n & \cos^{n-2}\varphi_n & \dots & \cos\varphi_n & 1 \end{vmatrix}$.

260 $\begin{vmatrix} 1 & 1 & \dots & 1 \\ \sin\varphi_1 & \sin\varphi_2 & \dots & \sin\varphi_n \\ \sin^2\varphi_1 & \sin^2\varphi_2 & \dots & \sin^2\varphi_n \\ \cdot & \cdot & \cdot & \cdot \\ \sin^{n-1}\varphi_1 & \sin^{n-1}\varphi_2 & \dots & \sin^{n-1}\varphi_n \end{vmatrix}$.

261 $\begin{vmatrix} a^n & (a-1)^n & \dots & (a-n)^n \\ a^{n-1} & (a-1)^{n-1} & \dots & (a-n)^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ a & a-1 & \dots & a-n \\ 1 & 1 & \dots & 1 \end{vmatrix}$.

262 $\begin{vmatrix} (a_1+x)^n & (a_1+x)^{n-1} & \dots & a_1+x & 1 \\ (a_2+x)^n & (a_2+x)^{n-1} & \dots & a_2+x & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a_{n+1}+x)^n & (a_{n+1}+x)^{n-1} & \dots & a_{n+1}+x & 1 \end{vmatrix}$.

263 $\begin{vmatrix} (2n-1)^n & (2n-2)^n & \dots & n^n & (2n)^n \\ (2n-1)^{n-1} & (2n-2)^{n-1} & \dots & n^{n-1} & (2n)^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2n-1 & 2n-2 & \dots & n & 2n \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix}$.

***264** $\begin{vmatrix} w_1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ w_2 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ w_n & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix}$.

***265**

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1+1 & x_2+1 & x_3+1 & \dots & x_n+1 \\ x_1^2+x_1 & x_2^2+x_2 & x_3^2+x_3 & \dots & x_n^2+x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1}+x_1^{n-2} & x_2^{n-1}+x_2^{n-2} & x_3^{n-1}+x_3^{n-2} & \dots & x_n^{n-1}+x_n^{n-2} \end{bmatrix}.$$

266

$$\begin{bmatrix} 1 & 1 & & 1 \\ 1+\sin\varphi_1 & 1+\sin\varphi_2 & \dots & 1+\sin\varphi_n \\ \sin\varphi_1+\sin^2\varphi_1 & \sin\varphi_2+\sin^2\varphi_2 & \dots & \sin\varphi_n+\sin^2\varphi_n \\ \cdot & \cdot & \cdot & \cdot \\ \sin^{n-2}\varphi_1+\sin^{n-1}\varphi_1 & \sin^{n-2}\varphi_2+\sin^{n-1}\varphi_2 & \dots & \sin^{n-2}\varphi_n+\sin^{n-1}\varphi_n \end{bmatrix}.$$

267

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \varphi_1(x_1) & \varphi_1(x_2) & \dots & \varphi_1(x_n) \\ \varphi_2(x_1) & \varphi_2(x_2) & \dots & \varphi_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_{n-1}(x_1) & \varphi_{n-1}(x_2) & \dots & \varphi_{n-1}(x_n) \end{bmatrix},$$

where $\varphi_k(x) = x^k + a_{1k}x^{k-1} + \dots + a_{kk}$.

268

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ F_1(\cos\varphi_1) & F_1(\cos\varphi_2) & \dots & F_1(\cos\varphi_n) \\ F_2(\cos\varphi_1) & F_2(\cos\varphi_2) & \dots & F_2(\cos\varphi_n) \\ \cdot & \cdot & \cdot & \cdot \\ F_{n-1}(\cos\varphi_1) & F_{n-1}(\cos\varphi_2) & \dots & F_{n-1}(\cos\varphi_n) \end{bmatrix},$$

where $F_k(x) = a_{0k}x^k + a_{1k}x^{k-1} + \dots + a_{kk}$.

*269

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \binom{x_1}{1} & \binom{x_2}{1} & \dots & \binom{x_n}{1} \\ \binom{x_1}{2} & \binom{x_2}{2} & \dots & \binom{x_n}{2} \\ \dots & \dots & \dots & \dots \\ \binom{x_1}{n-4} & \binom{x_2}{n-1} & \dots & \binom{x_n}{n-1} \end{bmatrix},$$

where $\binom{x}{k} = \dfrac{x(x-1)\dots(x-k+1)}{1\cdot 2\dots k}$.

*270 If $a_1, a_2, \cdots, a_n$ are rational integers, show that the quantity $1^{n-1}\,2^{n-2}\cdots(n-1)$ is a factor of of the determinant of the matrix

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{bmatrix}$$

Calculate the determinants of the following matrices:

***271** $$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2^3 & 3^3 & \dots & n^3 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{2n-1} & 3^{2n-1} & \dots & n^{2n-1} \end{bmatrix}.$$

***272** $$\begin{bmatrix} \dfrac{x_1}{x_1-1} & \dfrac{x_2}{x_2-1} & \dots & \dfrac{x_n}{x_n-1} \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}$$

***273** $$\begin{bmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \dots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \dots & a_2b_2^{n-1} & b_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \dots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{bmatrix}.$$

274

$$\begin{bmatrix} \sin^{n-1}\alpha_1 & \sin^{n-2}\alpha_1\cos\alpha_1 & \dots & \sin\alpha_1\cos^{n-2}\alpha_1 & \cos^{n-1}\alpha_1 \\ \sin^{n-1}\alpha_2 & \sin^{n-2}\alpha_2\cos\alpha_2 & \dots & \sin\alpha_2\cos^{n-2}\alpha_2 & \cos^{n-1}\alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ \sin^{n-1}\alpha_n & \sin^{n-2}\alpha_n\cos\alpha_n & \dots & \sin\alpha_n\cos^{n-2}\alpha_n & \cos^{n-1}\alpha_n \end{bmatrix}.$$

***275**

$$\begin{bmatrix} a_1^{2n}+1 & a_1^{2n-1}+a_1 & a_1^{2n-2}+a_1^2 & \dots & a_1^{n+1}+a_1^{n-1} & a_1^n \\ a_2^{2n}+1 & a_2^{2n-1}+a_2 & a_2^{2n-2}+a_2^2 & \dots & a_2^{n+1}+a_2^{n-1} & a_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n+1}^{2n}+1 & a_{n+1}^{2n-1}+a_{n+1} & a_{n+1}^{2n-2}+a_{n+1}^2 & \dots & a_{n+1}^{n+1}+a_{n+1}^{n-1} & a_{n+1}^n \end{bmatrix}.$$

***276**

$$\begin{bmatrix} 1 & \cos\varphi_0 & \cos 2\varphi_0 & \dots & \cos(n-1)\varphi_0 \\ 1 & \cos\varphi_1 & \cos 2\varphi_1 & \dots & \cos(n-1)\varphi_1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos\varphi_{n-1} & \cos 2\varphi_{n-1} & \dots & \cos(n-1)\varphi_{n-1} \end{bmatrix}.$$

***277**

$$\begin{bmatrix} \sin(n+1)\alpha_0 & \sin n\alpha_0 & \dots & \sin\alpha_0 \\ \sin(n+1)\alpha_1 & \sin n\alpha_1 & \dots & \sin\alpha_1 \\ \dots & \dots & \dots & \dots \\ \sin(n+1)\alpha_n & \sin n\alpha_n & \dots & \sin\alpha_n \end{bmatrix}.$$

***278**

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1(x_1-1) & x_2(x_2-1) & \dots & x_n(x_n-1) \\ x_1^2(x_1-1) & x_2^2(x_2-1) & \dots & x_n^2(x_n-1) \\ \dots & \dots & \dots & \dots \\ x_1^{n-1}(x_1-1) & x_2^{n-1}(x_2-1) & \dots & x_n^{n-1}(x_n-1) \end{bmatrix}.$$

***279**

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ x_1^3 & x_2^3 & \dots & x_n^3 \\ \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n \end{bmatrix}.$$

***280**

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ x_1^n & x_2^n & \dots & x_n^n \end{bmatrix}.$$

281 $$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{s-1} & x_2^{s-1} & \dots & x_n^{s-1} \\ x_1^{s+1} & x_2^{s+1} & \dots & x_n^{s+1} \\ x_1^n & x_2^n & \dots & x_n^n \end{bmatrix}.$$

***282** $$\begin{bmatrix} 1+x_1 & 1+x_1^2 & \dots & 1+x_1^n \\ 1+x_2 & 1+x_2^2 & \dots & 1+x_2^n \\ \cdot & \cdot & \cdot & \cdot \\ 1+x_n & 1+x_n^2 & \dots & 1+x_n^n \end{bmatrix}.$$

283 $$\begin{bmatrix} 1 & x & x^2 & x^3 \\ x^3 & x^2 & x & 1 \\ 1 & 2x & 3x^2 & 4x^3 \\ 4x^3 & 3x^2 & 2x & 1 \end{bmatrix}.$$

284 $$\begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 & 5x^4 \\ 1 & 4x & 9x^2 & 16x^3 & 25x^4 \\ 1 & y & y^2 & y^3 & y^4 \\ 1 & 2y & 3y^2 & 4y^3 & 5y^4 \end{bmatrix}.$$

***285** $$\begin{bmatrix} 1 & x & x^2 & \dots & x^n \\ 1 & 2x & 3x^2 & \dots & (n+1)x^n \\ 1 & 2^2x & 3^2x^2 & \dots & (n+1)^2x^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2^{n-1}x & 3^{n-1}x^2 & \dots & (n+1)^{n-1}x^n \\ 1 & y & y^2 & \dots & y^n \end{bmatrix}.$$

***286** $$\begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & 2x & 3x^2 & \dots & nx^{n-1} \\ 1 & 2^2x & 3^2x^2 & \dots & n^2x^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2^{k-1}x & 3^{k-1}x^2 & \dots & n^{k-1}x^{n-1} \\ 1 & y_1 & y_1^2 & \dots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \dots & y_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & y_{n-k} & y_{n-k}^2 & \dots & y_{n-k}^{n-1} \end{bmatrix}.$$

***287** $$\begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 0 & 1 & C_1^2x & \dots & C_1^{n-1}x^{n-2} \\ 0 & 0 & 1 & \dots & C_2^{n-1}x^{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & C_{k-1}^{n-1}x^{n-k} \\ 1 & y & y^2 & \dots & y^{n-1} \\ 0 & 1 & C_1^2y & \dots & C_1^{n-1}y^{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & C_{n-k-1}^{n-1}y^k \end{bmatrix}.$$

288 a) Write down the Laplace expansion of the determinant of a 4-th order matrix by minors of the first two rows.

b) Expand the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix},$$

by some set of minors of order 2.

c) The same for the determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

d) Use the same method for the matrix of problem 145.

Calculate the determinants of the following matrices:

e) $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \\ 3 & 6 & 10 & 0 & 0 & 0 \\ 4 & 9 & 14 & 1 & 1 & 1 \\ 5 & 15 & 24 & 1 & 5 & 9 \\ 9 & 24 & 38 & 1 & 25 & 81 \end{bmatrix};$ f) $\begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & c_1 & 0 & d_1 \\ b_2 & 0 & a_2 & 0 \\ 0 & d_2 & 0 & c_2 \end{bmatrix};$

g) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ x_1 & x_2 & 0 & 0 & 0 & x_3 \\ a_1 & b_1 & 1 & 1 & 1 & c_1 \\ a_2 & b_2 & x_1 & x_2 & x_3 & c_2 \\ a_3 & b_3 & x_1^2 & x_2^2 & x_3^2 & c_3 \\ x_1^2 & x_2^2 & 0 & 0 & 0 & x_3^2 \end{bmatrix}$; h) $\begin{bmatrix} \lambda & 0 & 0 & \dots & 0 & a \\ x_1 & \alpha & \beta & \dots & \beta & y_1 \\ x_2 & \beta & \alpha & \dots & \beta & y_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n & \beta & \beta & \dots & \alpha & y_n \\ a & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}$.

i) Expand the determinant of the matrix of problem 230 by the method of Laplace.

j) The same for the matrix of problem 171.

k) Calculate the determinant of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix}.$$

l) Let A, B, C, D, be the determinants of the respective third order matrices obtained by omitting the first, second, third, and fourth columns in the following array.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

Establish the following identity:

$$\det\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \end{bmatrix} = AD - BC.$$

*m) A certain fifteenth order matrix has the form

$$\begin{bmatrix} \Delta & \Delta_1 & \Delta_1 \\ \Delta_1 & \Delta & \Delta_1 \\ \Delta_1 & \Delta_1 & \Delta \end{bmatrix},$$

where Δ, Δ_1 are given by

$$\Delta = \begin{pmatrix} a & x & x & -x & -x \\ x & 2a & a & 0 & 0 \\ x & a & 2a & 0 & 0 \\ -x & 0 & 0 & 2a & a \\ -x & 0 & 0 & a & 2a \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Calculate the determinant of the original matrix.

6. MULTIPLICATION OF DETERMINANTS

289 The following expressions are intended to represent products of determinants. Calculate the value of each expression by multiplying together a pair of matrices.

a) $$\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix};$$

b) $$\begin{vmatrix} 3 & 2 & 5 \\ -1 & 3 & 6 \\ 1 & -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} -2 & 3 & 4 \\ -1 & -3 & 5 \\ 2 & 1 & -1 \end{vmatrix};$$

c) $$\begin{vmatrix} 2 & 1 & 1 & 1 \\ -1 & 2 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & -1 & -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix}.$$

290 First multiply the matrices Δ, δ, and in that way compute the product $\det \Delta \cdot \det \delta$.

a) $$\Delta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & -3 & -8 \\ -1 & 1 & 0 & -13 \\ 2 & 3 & 5 & 15 \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 & -2 & -3 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

b) $$\Delta = \begin{bmatrix} -1 & -9 & -2 & 3 \\ -5 & 5 & 3 & -2 \\ -12 & -6 & 1 & 1 \\ 9 & 0 & -2 & 1 \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix};$$

c) $$\Delta = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

291 Square each of the following matrices and that way compute the value of the determinant of the original matrix:

$$\text{a) } \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}; \qquad \text{b) } \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 2 & 1 & 1 \\ 2 & 0 & -3 & -1 \\ 3 & -7 & -1 & 9 \end{bmatrix};$$

$$\text{c) } \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}.$$

292 Define D by

$$\det \begin{bmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n-1} \\ a_{10} & a_{11} & a_{12} & \dots & a_{1n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} \end{bmatrix} = D.$$

Let $\varphi_i(x) = a_{0i} + a_{1i}x + \dots + a_{n-1,i}x^{n-1}$

Find the value of

$$\det \begin{bmatrix} \varphi_0(x_1) & \varphi_0(x_2) & \dots & \varphi_0(x_n) \\ \varphi_1(x_1) & \varphi_1(x_2) & \dots & \varphi_1(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_{n-1}(x_1) & \varphi_{n-1}(x_2) & \dots & \varphi_{n-1}(x_n) \end{bmatrix}.$$

Show how this can be used to solve problems 265, 267, 268.

Calculate the determinants of the following matrices:

***293**

a)
$$\begin{bmatrix} (b_0+a_0)^n & (b_1+a_0)^n & \dots & (b_n+a_0)^n \\ (b_0+a_1)^n & (b_1+a_1)^n & \dots & (b_n+a_1)^n \\ \dots & \dots & \dots & \dots \\ (b_0+a_n)^n & (b_1+a_n)^n & \dots & (b_n+a_n)^n \end{bmatrix};$$

b)
$$\begin{bmatrix} \dfrac{1-\alpha_1^n\beta_1^n}{1-\alpha_1\beta_1} & \dfrac{1-\alpha_1^n\beta_2^n}{1-\alpha_1\beta_2} & \dots & \dfrac{1-\alpha_1^n\beta_n^n}{1-\alpha_1\beta_n} \\ \dfrac{1-\alpha_2^n\beta_1^n}{1-\alpha_2\beta_1} & \dfrac{1-\alpha_2^n\beta_2^n}{1-\alpha_2\beta_2} & \dots & \dfrac{1-\alpha_2^n\beta_n^n}{1-\alpha_2\beta_n} \\ \dots & \dots & \dots & \dots \\ \dfrac{1-\alpha_n^n\beta_1^n}{1-\alpha_n\beta_1} & \dfrac{1-\alpha_n^n\beta_2}{1-\alpha_n\beta_2} & \dots & \dfrac{1-\alpha_n^n\beta_n^n}{1-\alpha_n\beta_n} \end{bmatrix}.$$

***294**
$$\begin{bmatrix} \sin 2\alpha_1 & \sin(\alpha_1+\alpha_2) & \dots & \sin(\alpha_1+\alpha_n) \\ \sin(\alpha_2+\alpha_1) & \sin 2\alpha_2 & \dots & \sin(\alpha_2+\alpha_n) \\ \dots & \dots & \dots & \dots \\ \sin(\alpha_n+\alpha_1) & \sin(\alpha_n+\alpha_2) & \dots & \sin 2\alpha_n \end{bmatrix}.$$

***295**
$$\begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} & 1 \\ s_1 & s_2 & s_3 & \dots & s_n & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} & x^{n-1} \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n-1} & x^n \end{bmatrix},$$

where $s_k = x_1^k + x_2^k + \dots + x_n^k$.

*296

$$\begin{bmatrix} a & b & c & d & l & m & n & p \\ b & -a & -d & -c & m & -l & p & -n \\ c & d & -a & -b & n & -p & -l & m \\ d & -c & b & -a & p & n & -m & -l \\ l & -m & -n & -p & -a & b & c & d \\ m & l & p & -n & -b & -a & d & -c \\ n & -p & l & m & -c & -d & -a & b \\ p & n & -m & l & -d & c & -b & -a \end{bmatrix}.$$

***297**

$$\begin{bmatrix} \cos\varphi & \sin\varphi & \cos\varphi & \sin\varphi \\ \cos 2\varphi & \sin 2\varphi & 2\cos 2\varphi & 2\sin 2\varphi \\ \cos 3\varphi & \sin 3\varphi & 3\cos 3\varphi & 3\sin 3\varphi \\ \cos 4\varphi & \sin 4\varphi & 4\cos 4\varphi & 4\sin 4\varphi \end{bmatrix}.$$

***298**

$$\begin{bmatrix} \cos n\varphi & n\cos n\varphi & \sin n\varphi & n\sin n\varphi \\ \cos(n+1)\varphi & (n+1)\cos(n+1)\varphi & \sin(n+1)\varphi & (n+1)\sin(n+1)\varphi \\ \cos(n+2)\varphi & (n+2)\cos(n+2)\varphi & \sin(n+2)\varphi & (n+2)\sin(n+2)\varphi \\ \cos(n+3)\varphi & (n+3)\cos(n+3)\varphi & \sin(n+3)\varphi & (n+3)\sin(n+3)\varphi \end{bmatrix}.$$

*299

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{bmatrix},$$

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

*300

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \dots & a_0 \end{bmatrix}.$$

(circulant matrix)

301 Use the result of problem 300 to calculate the determinant of the particular circulant

$$\begin{bmatrix} x & u & z & y \\ y & x & u & z \\ z & y & x & u \\ u & z & y & x \end{bmatrix}.$$

302 Also use the results of problem 300 to solve problems 192, 205, 255.

Calculate the determinants of the following matrices:

303
$$\begin{bmatrix} 1 & C_1^{n-1} & C_2^{n-1} & \dots & C_{n-2}^{n-1} & 1 \\ 1 & 1 & C_1^{n-1} & \dots & C_{n-3}^{n-1} & C_{n-2}^{n-1} \\ C_{n-2}^{n-1} & 1 & 1 & \dots & C_{n-4}^{n-1} & C_{n-3}^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_1^{n-1} & C_2^{n-1} & C_3^{n-1} & \dots & 1 & 1 \end{bmatrix}.$$

304
$$\begin{bmatrix} 1 & 2a & 3a^2 & \dots & na^{n-1} \\ na^{n-1} & 1 & 2a & \dots & (n-1)a^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 2a & 3a^2 & 4a^3 & \dots & 1 \end{bmatrix}.$$

305
$$\begin{bmatrix} s-a_1 & s-a_2 & \dots & s-a_n \\ s-a_n & s-a_1 & \dots & s-a_{n-1} \\ \dots & \dots & \dots & \dots \\ s-a_2 & s-a_3 & \dots & s-a_1 \end{bmatrix},$$

where $s=a_1+a_2+\dots+a_n$.

306
$$\begin{bmatrix} t^{n-1} & C_1^n t^{n-2} & C_2^n t^{n-3} & \dots & C_{n-2}^n t & C_{n-1}^n \\ C_{n-1}^n & t^{n-1} & C_1^n t^{n-2} & \dots & C_{n-3}^n t^2 & C_{n-2}^n t \\ C_{n-2}^n t & C_{n-1}^n & t^{n-1} & \dots & C_{n-4}^n t^3 & C_{n-3}^n t^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_1^n t^{n-2} & C_2^n t^{n-3} & C_3^n t^{n-4} & \dots & C_{n-1}^n & t^{n-1} \end{bmatrix}.$$

307
$$\begin{bmatrix} \overbrace{-1 & -1 & \ldots & -1 & -1}^{p} & \overbrace{1 & 1 & \ldots & 1}^{n-p} \\ 1 & -1 & \ldots & -1 & -1 & -1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & -1 & -1 & -1 & -1 & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \ldots & -1 & 1 & 1 & 1 & \ldots & -1 \end{bmatrix}.$$

***308**
$$\begin{bmatrix} \cos\frac{\pi}{n} & \cos\frac{2\pi}{n} & \ldots & -1 \\ -1 & \cos\frac{\pi}{n} & \ldots & \cos\frac{(n-1)\pi}{n} \\ \cos\frac{(n-1)\pi}{n} & -1 & \ldots & \cos\frac{(n-2)\pi}{n} \\ \cdot & \cdot & \cdot & \cdot \\ \cos\frac{2\pi}{n} & \cos\frac{3\pi}{n} & \ldots & \cos\frac{\pi}{n} \end{bmatrix}.$$

309
$$\begin{bmatrix} \cos\theta & \cos 2\theta & \ldots & \cos n\theta \\ \cos n\theta & \cos\theta & \ldots & \cos(n-1)\theta \\ \cdot & \cdot & \cdot & \cdot \\ \cos 2\theta & \cos 3\theta & \ldots & \cos\theta \end{bmatrix}.$$

310
$$\begin{bmatrix} \sin a & \sin(a+h) & \sin(a+2h) & \ldots & \sin[a+(n-1)h] \\ \sin[a+(n-1)h] & \sin a & \sin(a+h) & \ldots & \sin[a+(n-2)h] \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sin(a+h) & \sin(a+2h) & \sin(a+3h) & \ldots & \sin a \end{bmatrix}.$$

***311**
$$\begin{bmatrix} 1^2 & 2^2 & 3^2 & \ldots & n^2 \\ n^2 & 1^2 & 2^2 & \ldots & (n-1)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2^2 & 3^2 & 4^2 & \ldots & 1^2 \end{bmatrix}.$$

312 Establish the following relation

$$\det\begin{bmatrix} a_0 & a_1 & a_1 & a_2 & a_1 & a_2 & a_2 \\ a_2 & a_0 & a_1 & a_1 & a_2 & a_1 & a_2 \\ a_2 & a_2 & a_0 & a_1 & a_1 & a_2 & a_1 \\ a_1 & a_2 & a_2 & a_0 & a_1 & a_1 & a_2 \\ a_2 & a_1 & a_2 & a_2 & a_0 & a_1 & a_1 \\ a_1 & a_2 & a_1 & a_2 & a_2 & a_0 & a_1 \\ a_1 & a_1 & a_2 & a_1 & a_2 & a_2 & a_0 \end{bmatrix} =$$

$$= (a_0 + 3a_1 + 3a_2)(a_0^2 - a_0a_1 - a_0a_2 + 2a_1^2 + 2a_2^2 - 3a_1a_2)^3.$$

313 Calculate the determinant of the following skew-circulant matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ -a_{n-1} & -a_n & a_1 & \dots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_2 & -a_3 & -a_4 & \dots & a_1 \end{bmatrix}$$

*314 Let A be a circulant matrix of order 2n. Show that det A can be written as a product of det B · det C where B is a circulant of order n and C is a skew-circulant of order n.

315 Calculate the determinant of the following matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ \mu a_n & a_1 & a_2 & \dots & a_{n-1} \\ \mu a_{n-1} & \mu a_n & a_1 & \dots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \mu a_2 & \mu a_3 & \mu a_4 & \dots & a_1 \end{bmatrix}.$$

7. MISCELLANEOUS PROBLEMS

316 Show that the derivative of the determinant of the matrix

$$M = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{bmatrix},$$

is given by the formula

$$(\det M)' = \Delta'(x) = \det \begin{bmatrix} a'_{11}(x) & a'_{12}(x) & \dots & a'_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{bmatrix} + \dots$$

$$\dots + \det \begin{bmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a'_{n1}(x) & a'_{n2}(x) & \dots & a'_{nn}(x) \end{bmatrix}.$$

317 Let A_{ik} be the algebraic co-factors of the elements a_{ik} in the matrix $[a_{ik}]$. Establish the relation

$$\det \begin{bmatrix} a_{11}+x & a_{12}+x & \dots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \dots & a_{2n}+x \\ \dots & \dots & \dots & \dots \\ a_{n1}+x & a_{n2}+x & \dots & a_{nn}+x \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} +$$

$$+ x \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik},$$

318 Show how the result of problem 317 can be used to compute the results of problems 200, 223, 224, 225, 226, 227, 228, 232, 233, 248, 249, and 250.

319 Show that the sum of the algebraic co-factors of all the elements of the square matrix $[a_{ij}]$ has the value

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2n}-a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}-a_{n-1,1} & a_{n2}-a_{n-1,2} & \cdots & a_{nn}-a_{n-1,n} \end{bmatrix}.$$

Establish the following theorems:

320 Let the matrix B be obtained from the matrix A by adding one and the same constant to every element of A. Show that the sum of the co-factors of the elements of A is the same as the sum of the co-factors of the elements of B.

321 Suppose all the elements of some row [column] of a matrix are unity. Show that the sum of the algebraic co-factors of the elements of the matrix is equal to the determinant of the matrix.

322 In problem 250 find the sum of all the algebraic co-factors of the elements of the matrix.

*323 Calculate the determinant of the matrix

$$\begin{bmatrix} (a_1+b_1)^{-1} & (a_1+b_2)^{-1} & \dots & (a_1+b_n)^{-1} \\ (a_2+b_1)^{-1} & (a_2+b_2)^{-1} & \dots & (a_2+b_n)^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ (a_n+b_1)^{-1} & (a_n+b_2)^{-1} & \dots & (a_n+b_n)^{-1} \end{bmatrix}.$$

324 Let P_n, Q_n represent the following two determinants

$$\det \begin{bmatrix} a_0 & 1 & 0 & \dots & 0 & 0 \\ -1 & a_1 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_{n-2} & 1 \\ 0 & 0 & 0 & \dots & -1 & a_{n-1} \end{bmatrix}, \quad \det \begin{bmatrix} a_1 & 1 & 0 & \dots & 0 & 0 \\ -1 & a_2 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_{n-2} & 1 \\ 0 & 0 & 0 & \dots & -1 & a_{n-1} \end{bmatrix}$$

Establish the relation

$$\frac{P_n}{Q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{n-1}}}}}.$$

Calculate the determinants of the following matrices:

*325
$$\begin{bmatrix} c & a & 0 & \dots & 0 & 0 \\ b & c & a & \dots & 0 & 0 \\ 0 & b & c & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & c & a \\ 0 & 0 & 0 & \dots & b & c \end{bmatrix}.$$

326
$$\begin{bmatrix} p & q & 0 & \dots & 0 & 0 \\ 2 & p & q & \dots & 0 & 0 \\ 0 & 1 & p & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & p & q \\ 0 & 0 & 0 & \dots & 1 & p \end{bmatrix}.$$

*327 Write as a polynomial in x the determinant of the following matrix

$$\begin{bmatrix} a_{11}+x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}+x & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn}+x \end{bmatrix}$$

*328 Calculate the determinate of the matrix of order 2n - 1 that has the following form. The first n - 1 elements of the principal diagonal are unity, the remaining elements of the principal diagonal are n. In the first n - 1 rows of the matrix there is a sequence of n 1's. In the remaining n rows of the matrix the elements to the left of the principal diagonal, reading from the principal diagonal leftward, are n - 1, n - 2,···, 1. All other elements of the matrix are 0. For example,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}; \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

Calculate the determinant of the following matrix:

*329

$$\begin{bmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ -n & x-2 & 2 & 0 & \dots & 0 & 0 \\ 0 & -(n-1) & x-4 & 3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -1, & x-2n \end{bmatrix}.$$

330

$$\begin{bmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ n-1 & x & 2 & 0 & \dots & 0 & 0 \\ 0 & n-2 & x & 3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & x \end{bmatrix}.$$

331
$$\begin{bmatrix} x & a & 0 & 0 & \dots & 0 & 0 \\ n(a-1) & x-1 & 2a & 0 & \dots & 0 & 0 \\ 0 & (n-1)(a-1) & x-2 & 3a & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a-1 & x-n \end{bmatrix}.$$

332
$$\begin{bmatrix} 1^{n-1} & 2^{n-1} & \dots & n^{n-1} \\ 2^{n-1} & 3^{n-1} & \dots & (n+1)^{n-1} \\ \dots & \dots & \dots & \dots \\ n^{n-1} & (n+1)^{n-1} & \dots & (2n-1)^{n-1} \end{bmatrix}.$$

333
$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & \frac{1}{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \dots & \frac{1}{2n-1} \end{bmatrix}.$$

334 Calculate the coefficient of the highest power of x in the determinant of the matrix

$$\begin{bmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & \dots & (1+x)^{a_1b_n} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & \dots & (1+x)^{a_2b_n} \\ \dots & \dots & \dots & \dots \\ (1+x)^{a_nb_1} & (1+x)^{a_nb_2} & \dots & (1+x)^{a_nb_n} \end{bmatrix}.$$

CHAPTER III - PROBLEMS

SYSTEMS OF LINEAR EQUATIONS

1. CRAMER'S RULE

Solve each of the following systems of equations:

335 $$\begin{aligned} 2x_1 - x_2 - x_3 &= 4, \\ 3x_1 + 4x_2 - 2x_3 &= 11, \\ 3x_1 - 2x_2 + 4x_3 &= 11. \end{aligned}$$

336 $$\begin{aligned} x_1 + x_2 + 2x_3 &= -1, \\ 2x_1 - x_2 + 2x_3 &= -4, \\ 4x_1 + x_2 + 4x_3 &= -2. \end{aligned}$$

337 $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5, \\ 2x_1 + 3x_2 + x_3 &= 1, \\ 2x_1 + x_2 + 3x_3 &= 11. \end{aligned}$$

338 $$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 31, \\ 5x_1 + x_2 + 2x_3 &= 29, \\ 3x_1 - x_2 + x_3 &= 10. \end{aligned}$$

339

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 &= 1, \\ 3x_1 - x_2 - x_3 - 2x_4 &= -4, \\ 2x_1 + 3x_2 - x_3 - x_4 &= -6, \\ x_1 + 2x_2 + 3x_3 - x_4 &= -4. \end{aligned}$$

340

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 2x_4 &= 6, \\ 2x_1 - x_2 - 2x_3 - 3x_4 &= 8, \\ 3x_1 + 2x_2 - x_3 + 2x_4 &= 4, \\ 2x_1 - 3x_2 + 2x_3 + x_4 &= -8. \end{aligned}$$

341

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5, \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= 1, \\ 3x_1 + 2x_2 + x_3 + 2x_4 &= 1, \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= -5. \end{aligned}$$

342

$$\begin{aligned} x_2 - 3x_3 + 4x_4 &= -5, \\ x_1 \qquad - 2x_3 + 3x_4 &= -4, \\ 3x_1 + 2x_2 \qquad - 5x_4 &= 12, \\ 4x_1 + 3x_2 - 5x_3 \qquad &= 5. \end{aligned}$$

343

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + 2x_4 &= 4, \\ 3x_1 + 3x_2 + 3x_3 + 2x_4 &= 6, \\ 3x_1 - x_2 - x_3 + 2x_4 &= 6, \\ 3x_1 - x_2 + 3x_3 - x_4 &= 6. \end{aligned}$$

344

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0, \\ x_1 + 3x_2 + 6x_3 + 10x_4 &= 0, \\ x_1 + 4x_2 + 10x_3 + 20x_4 &= 0. \end{aligned}$$

345

$$\begin{aligned} x_1+3x_2+5x_3+7x_4&=12,\\ 3x_1+5x_2+7x_3+x_4&=0,\\ 5x_1+7x_2+x_3+3x_4&=4,\\ 7x_1+x_2+3x_3+5x_4&=16. \end{aligned}$$

346

$$\begin{aligned} x_1+x_2+x_3+x_4+x_5&=0,\\ x_1-x_2+2x_3-2x_4+3x_5&=0,\\ x_1+x_2+4x_3+4x_4+9x_5&=0,\\ x_1-x_2+8x_3-8x_4+27x_5&=0,\\ x_1+x_2+16x_3+16x_4+81x_5&=0. \end{aligned}$$

347

$$\begin{aligned} x_1+2x_2+3x_3+4x_4&=0,\\ x_1+x_2+2x_3+3x_4&=0,\\ x_1+5x_2+x_3+2x_4&=0,\\ x_1+5x_2+5x_3+2x_4&=0. \end{aligned}$$

348

$$\begin{aligned} x_1+x_2+x_3+x_4\qquad&=0,\\ x_2+x_3+x_4+x_5&=0,\\ x_1+2x_2+3x_3\qquad&=2,\\ x_2+2x_3+3x_4\qquad&=-2,\\ x_3+2x_4+3x_5&=2. \end{aligned}$$

349

$$\begin{aligned} x_1+4x_2+6x_3+4x_4+x_5&=0,\\ x_1+x_2+4x_3+6x_4+4x_5&=0,\\ 4x_1+x_2+x_3+4x_4+6x_5&=0,\\ 6x_1+4x_2+x_3+x_4+4x_5&=0,\\ 4x_1+6x_2+4x_3+x_4+x_5&=0. \end{aligned}$$

350

$$\begin{aligned} 2x_1+x_2+x_3+x_4+x_5&=2,\\ x_1+2x_2+x_3+x_4+x_5&=0,\\ x_1+x_2+3x_3+x_4+x_5&=3,\\ x_1+x_2+x_3+4x_4+x_5&=-2,\\ x_1+x_2+x_3+x_4+5x_5&=5. \end{aligned}$$

351

$$\begin{aligned} x_1+2x_2+3x_3+4x_4+5x_5&=13,\\ 2x_1+x_2+2x_3+3x_4+4x_5&=10,\\ 2x_1+2x_2+x_3+2x_4+3x_5&=11,\\ 2x_1+2x_2+2x_3+x_4+2x_5&=6,\\ 2x_1+2x_2+2x_3+2x_4+x_5&=3. \end{aligned}$$

352

$$\begin{aligned} x_1+2x_2-3x_3+4x_4-x_5&=-1,\\ 2x_1-x_2+3x_3-4x_4+2x_5&=8,\\ 3x_1+x_2-x_3+2x_4-x_5&=3,\\ 4x_1+3x_2+4x_3+2x_4+2x_5&=-2,\\ x_1-x_2-x_3+2x_4-3x_5&=-3. \end{aligned}$$

353 Show that $x_1 = x_2 = x_3 = x_4 = 1$ is a solution of the system

$$\begin{aligned}
2x_1 - 3x_2 + 4x_3 - 3x_4 &= 0,\\
3x_1 - x_2 + 11x_3 - 13x_4 &= 0,\\
4x_1 + 5x_2 - 7x_3 - 2x_4 &= 0,\\
13x_1 - 25x_2 + x_3 + 11x_4 &= 0.
\end{aligned}$$

Calculate the determinant of the matrix of coefficients of this system.

354 Let a, b, c, d be real numbers, not all equal to zero. Show that the following system has just one solution

$$\begin{aligned}
ax + by + cz + dt &= 0,\\
bx - ay + dz - ct &= 0,\\
cx - dy - az + bt &= 0,\\
dx + cy - bz - at &= 0
\end{aligned}$$

355 If $\alpha \neq \beta$ find a solution of the following system of equations:

$$\begin{aligned}
\alpha x_1 + \alpha x_2 + \dots + \alpha x_{n-1} + \beta x_n &= a_n,\\
\alpha x_1 + \alpha x_2 + \dots + \beta x_{n-1} + \alpha x_n &= a_{n-1},\\
\dots\dots\dots\dots\dots\dots&\dots\dots\\
\beta x_1 + \alpha x_2 + \dots + \alpha x_{n-1} + \alpha x_n &= a_1,
\end{aligned}$$

356 If the numbers $b_1, b_2, \cdots, b_n, \beta_1, \beta_2, \cdots, \beta_n$ are 2n different numbers, find a solution of the following system:

$$\begin{gathered}
\frac{x_1}{b_1-\beta_1}+\frac{x_2}{b_1-\beta_2}+\ldots+\frac{x_n}{b_1-\beta_n}=1, \\
\frac{x_1}{b_2-\beta_1}+\frac{x_2}{b_2-\beta_2}+\ldots+\frac{x_n}{b_2-\beta_n}=1, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{x_1}{b_n-\beta_1}+\frac{x_2}{b_n-\beta_2}+\ldots+\frac{x_n}{b_n-\beta_n}=1,
\end{gathered}$$

357 If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all different, find a solution of the following system:

$$\begin{aligned}
x_1 &+ x_2 + \ldots + x_n = 1, \\
x_1\alpha_1 &+ x_2\alpha_2 + \ldots + x_n\alpha_n = t, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x_1\alpha_1^{n-1} &+ x_2\alpha_2^{n-1} + \ldots + x_n\alpha_n^{n-1} = t^{n-1},
\end{aligned}$$

358 If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all different, find a solution of the following system:

$$\begin{gathered}
x_1+x_2\alpha_1+\ldots+x_n\alpha_1^{n-1}=u_1, \\
x_1+x_2\alpha_2+\ldots+x_n\alpha_2^{n-1}=u_2, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
x_1+x_2\alpha_n+\ldots+x_n\alpha_n^{n-1}=u_n.
\end{gathered}$$

359 If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all different, find a solution of the following system:

$$\begin{aligned} x_1 + x_2 + \ldots + x_n &= u_1, \\ x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n &= u_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_1\alpha_1^{n-1} + x_2\alpha_2^{n-1} + \ldots + x_n\alpha_n^{n-1} &= u_n. \end{aligned}$$

360 Solve the following system of equations:

$$\begin{aligned} 1 + x_1 + x_2 + \ldots + x_n &= 0, \\ 1 + 2x_1 + 2^2x_2 + \ldots + 2^n x_n &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1 + nx_1 + n^2x_2 + \ldots + n^n x_n &= 0. \end{aligned}$$

2. THE RANK OF A MATRIX

361 How many k-th order submatrices are there in a matrix of m rows and n columns?

362 Construct matrices of rank a) 2; b) 3.

363 Show that the following operations do not change the rank of a matrix:

a) interchange of rows and columns;

b) multiplication of all the elements of a row [column] by a non-zero quantity;

c) interchange of two rows [columns];

d) addition to the elements of one row [column] the corresponding elements of another row [column] multiplied by an arbitrary quantity.

364 If two matrices A, B have the same number of rows and columns, the matrix obtained by adding corresponding elements of A and B is called the sum of the two matrices: C = A + B. Show that the rank of the sum of two matrices cannot exceed the sum of the ranks of the individual matrices.

365 How is the rank of the matrix altered if it is bordered by a) 1 additional column; b) 2 additional columns?

Calculate the rank of each of the following matrices:

366 $\begin{pmatrix} 0 & 4 & 10 & 1 \\ 4 & 8 & 18 & 7 \\ 10 & 18 & 40 & 17 \\ 1 & 7 & 17 & 3 \end{pmatrix}.$ **367** $\begin{pmatrix} 75 & 0 & 116 & -39 & 0 \\ 171 & -69 & 402 & 123 & 45 \\ 301 & 0 & 87 & -417 & -169 \\ 114 & -46 & 268 & 82 & 30 \end{pmatrix}.$

368 $\begin{pmatrix} 2 & 1 & 11 & 2 \\ 1 & 0 & 4 & -1 \\ 11 & 4 & 56 & 5 \\ 2 & -1 & 5 & -6 \end{pmatrix}.$ **369** $\begin{pmatrix} 14 & 12 & 6 & 8 & 2 \\ 6 & 104 & 21 & 9 & 17 \\ 7 & 6 & 3 & 4 & 1 \\ 35 & 30 & 15 & 20 & 5 \end{pmatrix}.$

370 $\begin{bmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{bmatrix}.$ **371** $\begin{bmatrix} 1 & -2 & 3 & -1 & -1 & -2 \\ 2 & -1 & 1 & 0 & -2 & -2 \\ -2 & -5 & 8 & -4 & 3 & -1 \\ 6 & 0 & -1 & 2 & -7 & -5 \\ -1 & -1 & 1 & -1 & 2 & 1 \end{bmatrix}.$

372 $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$ **373** $\begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & -5 & -12 \\ 3 & -7 & 8 & 9 & 13 \end{bmatrix}.$ **374** $\begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix}.$

375 $\begin{Vmatrix} 3 & 2 & -1 & 2 & 0 & 1 \\ 4 & 1 & 0 & -3 & 0 & 2 \\ 2 & -1 & -2 & 1 & 1 & -3 \\ 3 & 1 & 3 & -9 & -1 & 6 \\ 3 & -1 & -5 & 7 & 2 & -7 \end{Vmatrix}.$

376 $\begin{Vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{Vmatrix}.$

377 $\begin{Vmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 1 & 0 & -1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 \end{Vmatrix}.$

378 $\begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{Vmatrix}.$

379 $\begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$

380 $\begin{Vmatrix} 2 & -1 & 1 & 3 & 4 \\ 2 & -1 & 2 & 1 & -2 \\ 2 & -3 & 1 & 2 & -2 \\ 1 & 0 & 1 & -2 & -6 \\ 1 & 2 & 1 & -1 & 0 \\ 4 & -1 & 3 & -1 & -8 \end{Vmatrix}.$

3. SYSTEMS OF LINEAR FORMS

381 a) Give an example of two independent linear forms;

b) Give an example of three independent linear forms.

382 Give a system of four linear forms in five indeterminates, such that two of the forms are independent and the remaining are linear combinations of these two.

Find fundamental linear relations among the following systems of forms:

383

$$\begin{aligned} y_1 &= 2x_1 + 2x_2 + 7x_3 - x_4, \\ y_2 &= 3x_1 - x_2 + 2x_3 + 4x_4, \\ y_3 &= x_1 + x_2 + 3x_3 + x_4. \end{aligned}$$

384

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 - 5x_3 + 4x_4, \\ y_2 &= 3x_1 - x_2 + 3x_3 - 3x_4, \\ y_3 &= 3x_1 + 5x_2 - 13x_3 + 11x_4. \end{aligned}$$

385

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 - 4x_3 - x_4, \\ y_2 &= x_1 - 2x_2 + x_3 + 3x_4, \\ y_3 &= 5x_1 - 3x_2 - x_3 + 8x_4, \\ y_4 &= 3x_1 + 8x_2 - 9x_3 - 5x_4. \end{aligned}$$

386

$$\begin{aligned} y_1 &= 2x_1 + x_2 - x_3 + x_4, \\ y_2 &= x_1 + 2x_2 + x_3 - x_4, \\ y_3 &= x_1 + x_2 + 2x_3 + x_4. \end{aligned}$$

387

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 + x_4, \\ y_2 &= 2x_1 + 3x_2 + x_3 + 2x_4, \\ y_3 &= 3x_1 + x_2 + 2x_3 - 2x_4, \\ y_4 &= 4x_2 + 2x_3 + 5x_4. \end{aligned}$$

388

$$\begin{aligned} y_1 &= 2x_1 + x_2, \\ y_2 &= 3x_1 + 2x_2, \\ y_3 &= x_1 + x_2, \\ y_4 &= 2x_1 + 3x_2. \end{aligned}$$

389

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 + x_4 + x_5, \\ y_2 &= x_1 + 2x_2 + 3x_3 + 4x_4 + x_5, \\ y_3 &= x_1 + 3x_2 + 6x_3 + 10x_4 + x_5, \\ y_4 &= x_1 + 4x_2 + 10x_3 + 20x_4 + x_5. \end{aligned}$$

390

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 - 4x_4, \\ y_2 &= 2x_1 - x_2 + 2x_3 + 5x_4, \\ y_3 &= 2x_1 - x_2 + 5x_3 - 4x_4, \\ y_4 &= 2x_1 + 3x_2 - 4x_3 + x_4. \end{aligned}$$

391

$$\begin{aligned} y_1 &= 2x_1 + x_2 - 3x_3, \\ y_2 &= 3x_1 + x_2 - 5x_3, \\ y_3 &= 4x_1 + 2x_2 - x_3, \\ y_4 &= x_1 - 7x_3. \end{aligned}$$

392
$$\begin{aligned}
y_1 &= 2x_1 + 3x_2 + 5x_3 - 4x_4 + x_5,\\
y_2 &= x_1 - x_2 + 2x_3 + 3x_4 + 5x_5,\\
y_3 &= 3x_1 + 7x_2 + 8x_3 - 11x_4 - 3x_5,\\
y_4 &= x_1 - x_2 + x_3 - 2x_4 + 3x_5.
\end{aligned}$$

393
$$\begin{aligned}
y_1 &= 2x_1 - x_2 + 3x_3 + 4x_4 - x_5,\\
y_2 &= x_1 + 2x_2 - 3x_3 + x_4 + 2x_5,\\
y_3 &= 5x_1 - 5x_2 + 12x_3 + 11x_4 - 5x_5,\\
y_4 &= x_1 - 3x_2 + 6x_3 + 3x_4 - 3x_5.
\end{aligned}$$

394
$$\begin{aligned}
y_1 &= x_1 + 2x_2 + x_3 - 2x_4 + x_5,\\
y_2 &= 2x_1 - x_2 + x_3 + 3x_4 + 2x_5,\\
y_3 &= x_1 - x_2 + 2x_3 - x_4 + 3x_5,\\
y_4 &= 2x_1 + x_2 - 3x_3 + x_4 - 2x_5,\\
y_5 &= x_1 - x_2 + 3x_3 - x_4 + 7x_5.
\end{aligned}$$

395
$$\begin{aligned}
y_1 &= 4x_1 + 3x_2 - x_3 + x_4 - x_5,\\
y_2 &= 2x_1 + x_2 - 3x_3 + 2x_4 - 5x_5,\\
y_3 &= x_1 - 3x_2 \qquad + x_4 - 2x_5,\\
y_4 &= x_1 + 5x_2 + 2x_3 - 2x_4 + 6x_5.
\end{aligned}$$

396
$$\begin{aligned}
y_1 &= x_1 + 2x_2 - x_3 + 3x_4 - x_5 + 2x_6,\\
y_2 &= 2x_1 - x_2 + 3x_3 - 4x_4 + x_5 - x_6,\\
y_3 &= 3x_1 + x_2 - x_3 + 2x_4 + x_5 + 3x_6,\\
y_4 &= 4x_1 - 7x_2 + 8x_3 - 15x_4 + 6x_5 - 5x_6,\\
y_5 &= 5x_1 + 5x_2 - 6x_3 + 11x_4 \qquad + 9x_6.
\end{aligned}$$

397 In the following system, find what value λ must have so that the fourth form is a linear combination of the first three.

$$\begin{aligned}
y_1 &= x_1 + 2x_2 + x_3 - 3x_4 + 2x_5, \\
y_2 &= 2x_1 + x_2 + x_3 + x_4 - 3x_5, \\
y_3 &= x_1 + x_2 + 2x_3 + 2x_4 - 2x_5, \\
y_4 &= 2x_1 + 3x_2 - 5x_3 - 17x_4 + \lambda x_5.
\end{aligned}$$

4. SYSTEMS OF LINEAR EQUATIONS

398 Solve the following system of equations:

$$\begin{aligned}
x_1 - 2x_2 + x_3 + x_4 &= 1, \\
x_1 - 2x_2 + x_3 - x_4 &= -1, \\
x_1 - 2x_2 + x_3 + 5x_4 &= 5.
\end{aligned}$$

399 For what value of λ does the following system of equations have a solution:

$$\begin{aligned}
2x_1 - x_2 + x_3 + x_4 &= 1, \\
x_1 + 2x_2 - x_3 + 4x_4 &= 2, \\
x_1 + 7x_2 - 4x_3 + 11x_4 &= \lambda.
\end{aligned}$$

Solve the following systems of equations:

400

$$\begin{aligned}
x_1 + x_2 - 3x_3 &= -1, \\
2x_1 + x_2 - 2x_3 &= 1, \\
x_1 + x_2 + x_3 &= 3, \\
x_1 + 2x_2 - 3x_3 &= 1.
\end{aligned}$$

401

$$\begin{aligned}
2x_1 + x_2 + x_3 &= 2, \\
x_1 + 3x_2 + x_3 &= 5, \\
x_1 + x_2 + 5x_3 &= -7, \\
2x_1 + 3x_2 - 3x_3 &= 14.
\end{aligned}$$

402
$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 3, \\ 3x_1 + x_2 - 5x_3 &= 0, \\ 4x_1 - x_2 + x_3 &= 3, \\ x_1 + 3x_2 - 13x_3 &= -6. \end{aligned}$$

403
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 0, \\ 2x_1 - x_2 + 3x_3 &= 0, \\ 3x_1 - 5x_2 + 4x_3 &= 0, \\ x_1 + 17x_2 + 4x_3 &= 0. \end{aligned}$$

404
$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= 1, \\ 3x_1 - 2x_2 + 2x_3 - 3x_4 &= 2, \\ 5x_1 + x_2 - x_3 + 2x_4 &= -1, \\ 2x_1 - x_2 + x_3 - 3x_4 &= 4. \end{aligned}$$

405
$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 1, \\ 2x_1 - x_2 - 3x_4 &= 2, \\ 3x_1 - x_3 + x_4 &= -3, \\ 2x_1 + 2x_2 - 2x_3 + 5x_4 &= -6. \end{aligned}$$

406
$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 4, \\ x_2 - x_3 + x_4 &= -3, \\ x_1 + 3x_2 - 3x_4 &= 1, \\ -7x_2 + 3x_3 + x_4 &= -3. \end{aligned}$$

407
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 11, \\ 2x_1 + 3x_2 + 4x_3 + x_4 &= 12, \\ 3x_1 + 4x_2 + x_3 + 2x_4 &= 13, \\ 4x_1 + x_2 + 2x_3 + 3x_4 &= 14. \end{aligned}$$

408
$$\begin{aligned} 2x_1 + 3x_2 - x_3 + 5x_4 &= 0, \\ 3x_1 - x_2 + 2x_3 - 7x_4 &= 0, \\ 4x_1 + x_2 - 3x_3 + 6x_4 &= 0, \\ x_1 - 2x_2 + 4x_3 - 7x_4 &= 0. \end{aligned}$$

409
$$\begin{aligned} 3x_1 + 4x_2 - 5x_3 + 7x_4 &= 0, \\ 2x_1 - 3x_2 + 3x_3 - 2x_4 &= 0, \\ 4x_1 + 11x_2 - 13x_3 + 16x_4 &= 0, \\ 7x_1 - 2x_2 + x_3 + 3x_4 &= 0. \end{aligned}$$

410
$$\begin{aligned} x_1 + x_2 - 3x_4 - x_5 &= 0, \\ x_1 - x_2 + 2x_3 - x_4 &= 0, \\ 4x_1 - 2x_2 + 6x_3 + 3x_4 - 4x_5 &= 0, \\ 2x_1 + 4x_2 - 2x_3 + 4x_4 - 7x_5 &= 0. \end{aligned}$$

411
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 7, \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 &= -2, \\ x_2 + 2x_3 + 2x_4 + 6x_5 &= 23, \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 &= 12. \end{aligned}$$

412
$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 + x_5 &= 0, \\ 2x_1 + x_2 - x_3 + 2x_4 - 3x_5 &= 0, \\ 3x_1 - 2x_2 - x_3 + x_4 - 2x_5 &= 0, \\ 2x_1 - 5x_2 + x_3 - 2x_4 + 2x_5 &= 0. \end{aligned}$$

413
$$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 - x_5 &= 0,\\ 2x_1 + x_2 - x_3 - x_4 + x_5 &= 0,\\ x_1 + 7x_2 - 5x_3 - 5x_4 + 5x_5 &= 0,\\ 3x_1 - x_2 - 2x_3 + x_4 - x_5 &= 0. \end{aligned}$$

414
$$\begin{aligned} 2x_1 + x_2 - x_3 - x_4 + x_5 &= 1,\\ x_1 - x_2 + x_3 + x_4 - 2x_5 &= 0,\\ 3x_1 + 3x_2 - 3x_3 - 3x_4 + 4x_5 &= 2,\\ 4x_1 + 5x_2 - 5x_3 - 5x_4 + 7x_5 &= 3. \end{aligned}$$

415
$$\begin{aligned} 2x_1 - 2x_2 + x_3 - x_4 + x_5 &= 1,\\ x_1 + 2x_2 - x_3 + x_4 - 2x_5 &= 1,\\ 4x_1 - 10x_2 + 5x_3 - 5x_4 + 7x_5 &= 1,\\ 2x_1 - 14x_2 + 7x_3 - 7x_4 + 11x_5 &= -1. \end{aligned}$$

416
$$\begin{aligned} 3x_1 + x_2 - 2x_3 + x_4 - x_5 &= 1,\\ 2x_1 - x_2 + 7x_3 - 3x_4 + 5x_5 &= 2,\\ x_1 + 3x_2 - 2x_3 + 5x_4 - 7x_5 &= 3,\\ 3x_1 - 2x_2 + 7x_3 - 5x_4 + 8x_5 &= 3. \end{aligned}$$

417
$$\begin{aligned} x_1 + 2x_2 \phantom{{}+4x_3} - 3x_4 + 2x_5 &= 1,\\ x_1 - x_2 - 3x_3 + x_4 - 3x_5 &= 2,\\ 2x_1 - 3x_2 + 4x_3 - 5x_4 + 2x_5 &= 7,\\ 9x_1 - 9x_2 + 6x_3 - 16x_4 + 2x_5 &= 25. \end{aligned}$$

418
$$\begin{aligned} x_1 + 3x_2 + 5x_3 - 4x_4 \phantom{{}+x_5} &= 1,\\ x_1 + 3x_2 + 2x_3 - 2x_4 + x_5 &= -1,\\ x_1 - 2x_2 + x_3 - x_4 - x_5 &= 3,\\ x_1 - 4x_2 + x_3 + x_4 - x_5 &= 3,\\ x_1 + 2x_2 + x_3 - x_4 + x_5 &= -1, \end{aligned}$$

419
$$\begin{aligned} x_1 + 2x_2 + 3x_3 - x_4 &= 1,\\ 3x_1 + 2x_2 + x_3 - x_4 &= 1,\\ 2x_1 + 3x_2 + x_3 + x_4 &= 1,\\ 2x_1 + 2x_2 + 2x_3 - x_4 &= 1,\\ 5x_1 + 5x_2 + 2x_3 \phantom{{}-x_4} &= 2. \end{aligned}$$

420

$$\begin{aligned}
x_1 - 2x_2 + 3x_3 - 4x_4 + 2x_5 &= -2,\\
x_1 + 2x_2 - x_3 \phantom{{}-4x_4} - x_5 &= -3,\\
x_1 - x_2 + 2x_3 - 3x_4 \phantom{{}+2x_5} &= 10,\\
x_2 - x_3 + x_4 - 2x_5 &= -5,\\
2x_1 + 3x_2 - x_3 + x_4 + 4x_5 &= 1.
\end{aligned}$$

421 If it is known that the system of equations

$$\begin{aligned}
ay + bx &= c,\\
cx + az &= b,\\
bz + cy &= a
\end{aligned}$$

has a single solution, show that $abc \neq 0$ and find the solution itself.

Solve the following systems of equations:

422 $$\begin{aligned}
\lambda x + y + z &= 1,\\
x + \lambda y + z &= \lambda,\\
x + y + \lambda z &= \lambda^2.
\end{aligned}$$

423 $$\begin{aligned}
\lambda x + y + z + t &= 1,\\
x + \lambda y + z + t &= \lambda,\\
x + y + \lambda z + t &= \lambda^2,\\
x + y + z + \lambda t &= \lambda^3.
\end{aligned}$$

424 $$\begin{aligned}
x + ay + a^2 z &= a^3,\\
x + by + b^2 z &= b^3,\\
x + cy + c^2 z &= c^3.
\end{aligned}$$

425 $$\begin{aligned}
x + y + z &= 1,\\
ax + by + cz &= d,\\
a^2 x + b^2 y + c^2 z &= d^2.
\end{aligned}$$

426 $$\begin{aligned}
ax + y + z &= 4,\\
x + by + z &= 3,\\
x + 2by + z &= 4.
\end{aligned}$$

427 $$\begin{aligned}
ax + by + z &= 1,\\
x + aby + z &= b,\\
x + by + az &= 1.
\end{aligned}$$

428 $$\begin{aligned}
\alpha x + y + z &= m,\\
x + \alpha y + z &= n,\\
x + y + \alpha z &= p.
\end{aligned}$$

429 $$\begin{aligned}
x + ay + a^2 z &= 1,\\
x + ay + abz &= a,\\
bx + a^2 y + a^2 bz &= a^2 b.
\end{aligned}$$

430

$$\begin{aligned}(\lambda+3)x+y+2z&=\lambda,\\ \lambda x+(\lambda-1)y+z&=2\lambda,\\ 3(\lambda+1)x+\lambda y+(\lambda+3)z&=3.\end{aligned}$$

431

$$\begin{aligned}\lambda x+\lambda y+(\lambda+1)z&=\lambda,\\ \lambda x+\lambda y+(\lambda-1)z&=\lambda,\\ (\lambda+1)x+\lambda y+(2\lambda+3)z&=1.\end{aligned}$$

432

$$\begin{aligned}3kx+(2k+1)y+(k+1)z&=k,\\ (2k-1)x+(2k-1)y+(k-2)z&=k+1,\\ (4k-1)x+3ky+2kz&=1.\end{aligned}$$

433

$$\begin{aligned}ax+by+2z&=1,\\ ax+(2b-1)y+3z&=1,\\ ax+by+(b+3)z&=2b-1.\end{aligned}$$

434 a)

$$\begin{aligned}3mx+(3m-7)y+(m-5)z&=m-1,\\ (2m-1)x+(4m-1)y+2mz&=m+1,\\ 4mx+(5m-7)y+(2m-5)z&=0.\end{aligned}$$

b)

$$\begin{aligned}(2m+1)x-my+(m+1)z&=m-1,\\ (m-2)x+(m-1)y+(m-2)z&=m,\\ (2m-1)x+(m-1)y+(2m-1)z&=m.\end{aligned}$$

c)

$$\begin{aligned}(5\lambda+1)x+2\lambda y+(4\lambda+1)z&=1+\lambda,\\ (4\lambda-1)x+(\lambda-1)y+(4\lambda-1)z&=-1,\\ 2(3\lambda+1)x+2\lambda y+(5\lambda+2)z&=2-\lambda.\end{aligned}$$

435 a)

$$\begin{aligned}(2c+1)x-cy-(c+1)z&=2c,\\ 3cx-(2c-1)y-(3c-1)z&=c+1,\\ (c+2)x-y-2cz&=2.\end{aligned}$$

b)

$$\begin{aligned}2(\lambda+1)x+3y+\lambda z&=\lambda+4,\\ (4\lambda-1)x+(\lambda+1)y+(2\lambda-1)z&=2\lambda+2,\\ (5\lambda-4)x+(\lambda+1)y+(3\lambda-4)z&=\lambda-1.\end{aligned}$$

c) $$\begin{aligned} dx+(2d-1)y+(d+2)z&=1,\\ (d-1)y+(d-3)z&=1+d,\\ dx+(3d-2)y+(3d+1)z&=2-d. \end{aligned}$$

d) $$\begin{aligned} (3a-1)x+2ay+(3a+1)z&=1,\\ 2ax+2ay+(3a+1)z&=a,\\ (a+1)x+(a+1)y+2(a+1)z&=a^2. \end{aligned}$$

436 Find the equation of the straight line containing the points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$.

437 Find a determinantal equation that is necessary and sufficient for the collinearity of the three points $M_1(x_1, y_1)$; $M_2(x_2, y_2)$; $M_3(x_3, y_3)$.

438 Find a determinantal equation sufficient for the concurrence of the three lines

$$a_1x+b_1y+c_1=0;\ a_2x+b_2y+c_2=0;\ a_3x+b_3y+c_3=0$$

439 Find a determinantal equation sufficient for the concyclicity of the four points

$$M_0(x_0, y_0);\ M_1(x_1, y_1);\ M_2(x_2, y_2);\ M_3(x_3, y_3)$$

(Several points are called concyclic if they all lie on the same circumference.)

440 Find the equation of the circle containing the three points $M_1(2, 1)$; $M_2(1, 2)$; $M_3(0, 1)$.

441 Find the equation of the second degree curve containing the five points $M_1(0, 0)$; $M_2(1, 0)$; $M_3(-1, 0)$; $M_4(1, 1)$; $M_5(-1, 1)$.

442 Find the equation of the third degree parabola that contains the four points $M_1(1, 0)$; $M_2(0, -1)$; $M_3(-1, -2)$; $M_4(2, 7)$.

443 Find the equation of the n-th degree parabola $y=a_0x^n+a_1x^{n-1} + \dots + a_n$ containing the $n + 1$ points $M_0(x_0, y_0)$; $M_1(x_1, y_1)$; $M_2(x_2, y_2)$; $\cdots$; $M_n(x_n, y_n)$.

444 Find a determinantal equation sufficient for the coplanarity of the points $M_1(x_1, y_1, z_1)$; $M_2(x_2, y_2, z_2)$; $M_3(x_3, y_3, z_3)$; $M_4(x_4, y_4, z_4)$.

445 Find the equation of the sphere containing the four points $M_1(1, 0, 0)$; $M_2(1, 1, 0)$; $M_3(1, 1, 1)$; $M_4(0, 1, 1)$.

446 Under what conditions will the n points $M_1(x_1, y_1)$; $M_2(x_2, y_2)$; $M_3(x_3, y_3)$; $\cdots$; $M_n(x_n, y_n)$ lie on a straight line?

447 Under what conditions will the n lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $\cdots$; $a_nx + b_ny + c_n = 0$ be concurrent?

448 What are the conditions in 3-dimensional space that the n points $M_1(x_1, y_1, z_1)$; $M_2(x_2, y_2, z_2)$; $\cdots$; $M_n(x_n, y_n, z_n)$ will lie in a plane; will lie on a line?

449 Under what conditions will the n planes $A_i x + B_i y + C_i z + D_i = 0$ $(i = 1, 2, \cdots, n)$ be concurrent at a point; under what conditions will they be concurrent in a line?

450 Find $x_1, x_2, \cdots, x_{n-1}$ if the following system of n equalities is satisfied:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1,n-1}x_{n-1} + a_{1n} &= 0, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2,n-1}x_{n-1} + a_{2n} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{n,n-1}x_{n-1} + a_{nn} &= 0. \end{aligned}$$

451 Let

$$\left.\begin{array}{llll} x_1^{(1)} = \alpha_{11}; & x_2^{(1)} = \alpha_{12}; & \ldots; & x_n^{(1)} = \alpha_{1n}, \\ x_1^{(2)} = \alpha_{21}; & x_2^{(2)} = \alpha_{22}; & \ldots; & x_n^{(2)} = \alpha_{2n}, \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(m)} = \alpha_{m1}; & x_2^{(m)} = \alpha_{m2}; & \ldots; & x_n^{(m)} = \alpha_{mn} \end{array}\right\} \qquad (1)$$

be m solutions of some system of linear homogeneous equations. These solutions are called linearly dependent if there exist constants $c_1, c_2, \cdots, c_m$, not all 0, such that the relations

$$c_1\alpha_{1i} + c_2\alpha_{2i} + \ldots + c_m\alpha_{mi} = 0 \qquad (2)$$
$$(i = 1, 2, \ldots, n).$$

hold. If no such system of relations holds except the system with

$$c_1 = c_2 = \cdots = c_m = 0,$$

the solutions are called linearly independent. The solutions themselves can be written as rows of a matrix. Thus the m solutions (1) can be written in the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} = A.$$

Show that if the matrix A has rank r, then the system (1) has r linearly independent solutions, and all other solutions appearing in the array (1) are linear combinations of these r solutions.

452 Show that if the rank of a system of m linear homogeneous equations and n indeterminates is r, then the system has $n - r$ linearly independent solutions and all other solutions of this system are linear combinations of these $n - r$ solutions.

Such a system of $n - r$ solutions is called a fundamental system of solutions.

453 Decide whether

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -2 & 3 & -2 & 0 \end{pmatrix}$$

is a fundamental system of solutions of the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0, \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 &= 0, \\ x_2 + 2x_3 + 2x_4 + 6x_5 &= 0, \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 &= 0. \end{aligned}$$

454 Find a fundamental system of solutions for the system of equations appearing in 453.

455 Is

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 4 & 0 & 0 & -6 & 2 \end{pmatrix}$$

a fundamental system of solutions of the system in problem 453?

456 Show that if A is a matrix of rank r representing in its rows a fundamental system of solutions of a set of linear homogeneous equations, and B is an arbitrary non-singular r-th order matrix, then the matrix BA is also a fundamental system of solutions of the same system of homogeneous equations.

457 (Converse to 456.) Show that if each of two matrices A, C of rank r represent in their rows a fundamental system of solutions, then there is a non-singular r-th order matrix B such that the relation A = BC holds.

458 Let

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rn} \end{pmatrix}$$

be a fundamental system of solutions of some system of homogeneous equations. Show that

$$\begin{aligned} x_1 &= c_1\alpha_{11} + c_2\alpha_{21} + \cdots + c_r\alpha_{r1}, \\ x_2 &= c_1\alpha_{12} + c_2\alpha_{22} + \cdots + c_r\alpha_{r2}, \\ &\cdots\cdots\cdots\cdots \\ x_n &= c_1\alpha_{1n} + c_2\alpha_{2n} + \cdots + c_r\alpha_{rn} \end{aligned}$$

is the general solution of this system of equations. This means that every solutions of the system can be written in the form above, for appropriate choice of $c_1, c_2, \cdots, c_r$, and conversely.

459 Find the general solution of the system in problem 453.

460 Decide whether (11 1 -7) is a fundamental system of solutions of the system in problem 403 and also find the general solution.

461 Find the general solution of the systems in problems 408, 409, 410, 412, 413.

462 Use the general solution of the system in problem 453 (notice the answer from problem 459). Note that $x_1 = -16$, $x_2 = 23$, $x_3 = x_4 = x_5 = 0$ is a particular solution of the system of problem 411. Give the general solution of the latter system.

463 Give the general solution of the systems in problems 406, 414, 415.

CHAPTER IV - PROBLEMS

MATRICES

1. OPERATIONS WITH SQUARE MATRICES

464 Form the products of the following matrices:

a) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$;

c) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$; d) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & -4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$;

e) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$; f) $\begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix}$.

465 Carry out the following operations:

a) $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}^2$; b) $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^3$; c) $\begin{pmatrix} 3 & 2 \\ -4 & -2 \end{pmatrix}^5$;

d) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$; e) $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}^n$.

*466 If α is a real number calculate

$$\lim_{n\to\infty}\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n$$

467 Assuming that AB are permutable, AB = BA, establish the following

a) $(A+B)^2 = A^2 + 2AB + B^2$;
b) $A^2 - B^2 = (A+B)(A-B)$;
c) $(A+B)^n = A^n + \frac{n}{1} A^{n-1}B + \dots + B^n$.

468 Calculate AB - BA in each of the cases:

a) $A=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$;

b) $A=\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}$.

469 For each of the matrices A, find another matrix commutative with A:

$$\text{a) } A=\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \quad \text{b) } A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \text{c) } A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

470 In the given cases find f(A):

$$\text{a) } f(x)=x^2-x-1, \quad A=\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix};$$

$$\text{b) } f(x)=x^2-5x+3, \quad A=\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}.$$

471 Let A be an arbitrary second order matrix

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Show that the following relation is satisfied when A is substituted for x:

$$x^2-(a+d)x+(ad-bc)=0.$$

472 Let A be an arbitrary square matrix. Show that there is a polynomial f(x) such that the relation f(A) = 0 holds; furthermore that there is a privileged polynomial of this type which divides evenly every other such polynomial.

*473 Show that there are no two matrices A, B such that the relation $AB - BA = E$ holds.

474 Let A be a matrix such that $A^k = 0$. Show that the following relation holds:

$$(E - A)^{-1} = E + A + A^2 + \cdots + A^{k-1}$$

475 If the square of a second order matrix is the null matrix, what form must the second order matrix have?

476 If the cube of a second order matrix is the null matrix, what form must the second order matrix have?

477 If the square of a second order matrix is the identity matrix, what form must the original matrix have?

478 Let A be some second order matrix. Find all second order square matrices X satisfying the relation $XA = 0$.

479 Let A be a given second order matrix. Find all second order square matrices X satisfying the relation $X^2 = A$.

480 Find the inverse of each of the following matrices:

a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$; b) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; c) $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$;

d) $A=\begin{pmatrix} 1 & 3 & -5 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; e) $A=\begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$;

f) $A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 \cdots & 1 \end{pmatrix}$; g) $A=\begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 1 & 3 & 4 \\ 2 & -1 & 2 & 3 \end{pmatrix}$;

h) $A=\begin{Vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ . & . & . & . & . \\ 1 & 1 & 1 & \dots & 0 \end{Vmatrix}$; i) $A=\begin{Vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2n-2} \\ . & . & . & . & . \\ 1 & \varepsilon^{n-1} & \varepsilon^{2n-2} & \dots & \varepsilon^{(n-1)^2} \end{Vmatrix}$,

where $\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$:

j) $A=\begin{Vmatrix} 2 & -1 & 0 & \dots & & 0 \\ -1 & 2 & -1 & \dots & & 0 \\ 0 & -1 & 2 & \dots & & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & -1 & 2 \end{Vmatrix}$;

k) $A=\begin{Vmatrix} 1 & 3 & 5 & 7 & \dots & 2n-1 \\ 2n-1 & 1 & 3 & 5 & \dots & 2n-3 \\ 2n-3 & 2n-1 & 1 & 3 & \dots & 2n-5 \\ . & . & . & . & . & . \\ 3 & 5 & 7 & 9 & \dots & 1 \end{Vmatrix}$;

l) $A=\begin{Vmatrix} 1 & 0 & 0 & \dots & 0 & c_1 \\ 0 & 1 & 0 & \dots & 0 & c_2 \\ 0 & 0 & 1 & \dots & 0 & c_3 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 & c_n \\ b_1 & b_2 & b_3 & \dots & b_n & a \end{Vmatrix}$;

m) $A=\begin{pmatrix} 1 & -x & 0 & \dots & & 0 \\ 0 & 1 & -x & \dots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & -x \\ a_0 & a_1 & a_2 & \dots & & a_n \end{pmatrix}$;

n) $A=\begin{pmatrix} 1+\frac{1}{\lambda_1} & 1 & 1 & \dots & 1 \\ 1 & 1+\frac{1}{\lambda_2} & 1 & \dots & 1 \\ 1 & 1 & 1+\frac{1}{\lambda_3} & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \dots & 1+\frac{1}{\lambda_n} \end{pmatrix}$.

o) If the inverse B^{-1} is known show how to find the inverse of the bordered matrix

$$\begin{pmatrix} B & U \\ V & a \end{pmatrix}.$$

481 Find the square matrix X in each of the following matrix equations:

a) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix}$;

b) $X \cdot \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{pmatrix}$;

c) $\begin{Vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{Vmatrix} \cdot X = \begin{Vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 2 \end{Vmatrix}$;

d) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix}$;

e) $\begin{Vmatrix} 1 & 1 & 1 & \dots & & 1 \\ -1 & 1 & 0 & \dots & & 0 \\ 0 & -1 & 1 & \dots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -1 & 1 \end{Vmatrix} \cdot X \cdot \begin{Vmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \dots & 1 \end{Vmatrix} =$

$$= \begin{Vmatrix} 1 & 0 & 0 & \dots & & 0 \\ 0 & 2 & -1 & \dots & & 0 \\ 0 & -1 & 2 & \dots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{Vmatrix};$$

f) $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$; g) $X \cdot \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

482 Show that the relation $A^{-1}B = BA^{-1}$ follows from the relation $AB = BA$.

483 Suppose $\varphi(x) = \dfrac{1 + x}{1 - x}$, $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Calculate $\varphi(A)$.

484 Find all square second order matrices with real elements whose cubes are the identity matrix.

485 Find all square second order matrices with real elements whose fourth powers are equal to the identity matrix.

486 Let a, b be real. Show that the complex numbers under addition and multiplication are isomorphic to the set of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

487 Show that the set of all second order matrices

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$ where a, b, c, d have all

possible real values fill out a ring with no zero divisors.

488 Show how to express the product

$(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2)$ as the sum of four squares of bilinear expressions.

489 Show that each of the following three operations on a matrix can be realized by multiplying the matrix on the left by non-singular matrix:

a) interchange of two rows;

b) addition to one row of a multiple of the respective elements of another row;

c) multiplication of the elements of a row by a non-zero factor.

Corresponding operations on the columns of the matrix can be realized by multiplication on the right by a non-singular matrix.

490 Show that if A is an arbitrary matrix, A can be written in the form $A = PRQ$, where P and Q are non-singular matrices and the matrix R is diagonal with 1's in the first r diagonal positions, and 0's elsewhere.

*491 Show that every matrix of real numbers can be written as the product ot matrices of the form $\mathbf{E} + \alpha e_{ik}$, where α is a real number, $\mathbf{E}$ is the identity matrix and e_{ik} is a matrix unit; that is the matrix that has 1 in the ik position and 0's elsewhere.

*492 Show that the rank of the product of two square matrices of order n is no less than $r_1 + r_2 - n$, where r_1 and r_2 are the ranks of the factors.

493 Show that every square matrix of rank 1 has the form

$$\begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \cdots & \lambda_1\mu_n \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \cdots & \lambda_2\mu_n \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_n\mu_1 & \lambda_n\mu_2 & \cdots & \lambda_n\mu_n \end{pmatrix}.$$

Can all λ's be 0?

*494 Find all third order matrices whose squares are equal to the null matrix.

*495 Find all third order matrices whose squares are equal to the identity matrix.

*496 Let A, B be rectangular matrices with the same number of rows. The symbol (A, B) denotes the matrix obtained by appending the columns of the matrix B to the columns of the matrix A. Establish the relation $\operatorname{rank}(A, B) \leq \operatorname{rank} A + \operatorname{rank} B$.

*497 Let A be a square n-th order matrix. Show that if $A^2 = E$, then $\operatorname{rank}(E + A) + \operatorname{rank}(E - A) = n$.

*498 If B is a diagonal matrix with entries +1 and -1, and $A = PBP^{-1}$, then A obviously satisfies the relation $A^2 = E$. Establish the converse of this assertion, that is, that every matrix satisfying $A^2 = E$ can be written in the form PBP^{-1}, where P is a non-singular matrix.

499 Let A be a square matrix of integers. Under what conditions is the inverse A^{-1} also an integral matrix?

500 Let A be an arbitrary square non-singular integral matrix. Show that A can be written in the form A = PR, where P is an integral matrix of determinant +1 or -1; and R is an integral triagonal matrix; that is a matrix whose subdiagonal elements are all 0, whose diagonal elements are all positive. Moreover, every element above the main diagonal of R is non-negative and less than the diagonal element on the same row.

*501 Let k be an arbitrary integer. Partition the integral n-th order square matrices into equivalence classes, so that two elements belong to the same equivalence class if one can be obtained from the other by multiplication by a unimodular integral matrix (of determinant +1 or -1). How many equivalence classes are there?

502 Let A be an integral matrix. Show that A can be written in the form A = PRQ, where P, Q are integral unimodular matrices, and R is an integral diagonal matrix.

503 Show that every properly unimodular integral second order square matrix, that is every such matrix of determinant +1, can be written as a product of positive and negative powers of the fundamental generating matrices

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad B=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

504 Show that every second order integral unimodular matrix can be written as a product of positive and negative powers of the fundamental generating matrices

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

505 Let A be an arbitrary third order integral matrix with positive determinant, $A \neq E$, $A^2 = E$. Show that A can be written in the form QCQ^{-1}, where Q is an integral unimodular matrix, and C has one of the forms

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix},\quad \begin{pmatrix}1 & -1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

2. RECTANGULAR MATRICES. CERTAIN INEQUALITIES

506 Multiply the two matrices in each part:

a) $\begin{pmatrix}2 & 1 & 1\\ 3 & 0 & 1\end{pmatrix}$, $\begin{pmatrix}3 & 1\\ 2 & 1\\ 1 & 0\end{pmatrix}$; b) $\begin{pmatrix}3 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}$, $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$;

c) $\begin{pmatrix}2\\ 1\\ 3\end{pmatrix}$, $(1\ \ 2\ \ 3)$; d) $(1\ \ 2\ \ 3)$, $\begin{pmatrix}2\\ 4\\ 1\end{pmatrix}$.

507 Multiply the matrix

$$\begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$$

by its transpose and find the determinant of the product.

508 Multiply the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

by its transpose and apply the theorem on the determinant of a product to obtain an algebraic identity connecting the determinants.

509 Show how an m-th order minor of the product of two matrices is related to the minors of the factors.

510 Let A be an arbitrary matrix of real numbers, A' be its transpose. Show that every principal minor of the matrix A'A is non-negative.

511 Let A be a matrix of real numbers, and A' be its transpose. Show that the rank of A'A, as well as the rank of A, is less than k if every k-th order principal minor of the matrix A'A is zero.

512 Let A be an arbitrary matrix, A' be its transpose. Show that the sum of all principal k-th order minors of the matrix A'A is the same as the corresponding sum of the matrix AA'.

513 Choose rectangular matrices properly (see exercise 508); multiply them, and establish the identity

$$(a_1^2+a_2^2+\dots+a_n^2)(b_1^2+b_2^2+\dots+b_n^2)-$$
$$-(a_1b_1+a_2b_2+\dots+a_nb_n)^2=\sum_{i<k}(a_ib_k-a_kb_i)^2.$$

514 Let a_i, b_i be complex numbers; $\overline{b_i}$ be the complex conjugate of b_i. Establish the following identity.

$$\sum_{i=1}^{n}|a_i|^2\cdot\sum_{i=1}^{n}|b_i|^2-\left|\sum_{i=1}^{n}a_i\overline{b}_i\right|^2=\sum_{i<k}|a_ib_k-a_kb_i|^2.$$

515 Use the identity of problem 513 to establish the Buniakovsky inequality

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\leqslant\sum_{i=1}^{n}a_i^2\cdot\sum_{i=1}^{n}b_i^2$$

for arbitrary real a_i, b_i.

516 Establish the corresponding inequality

$$\left|\sum_{i=1}^{n}a_i\overline{b}_i\right|^2\leqslant\sum_{i=1}^{n}|a_i|^2\cdot\sum_{i=1}^{n}|b_i|^2$$

where a_i, b_i are arbitrary complex numbers.

*517 Let B, C be two real rectangular matrices; suppose that (B, C) = A is a square matrix. (This means that B is the matrix consisting of the first columns of A.) Show that $(\det A)^2 \leq (\det BB')(\det CC')$.

*518 If A is not square the generalization of 517 is as follows:

$$\det A'A \leq (\det B'B)(\det C'C).$$

519 Let A be the rectangular real matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Show that $\det AA' \leqslant \sum_{k=1}^{n} a_{1k}^2 \cdot \sum_{k=1}^{n} a_{2k}^2 \dots \sum_{k=1}^{n} a_{mk}^2$.

520 Let A be a rectangular matrix of complex numbers, A^* be the conjugate transpose matrix. Show that the determinant of A^*A is real, non-negative, and can be zero only if the rank of A is less than the number of columns it contains.

521 (Generalization of 518.) Suppose A = (B, C). Establish the identity

$$\det(A^*A) \leq \det(B^*B) \cdot \det(C^*C).$$

522 Let each element of the n-th order square matrix $A = [a_{ij}]$ have modulus less than M:

$$|a_{ik}| \leq M.$$

Establish the identity

$$|\det A| \leq M^n n^{n/2} .$$

*523 The following identity is a little harder to establish; it applies only when the elements are real.

$$|\det A| \leq M^n 2^{-n}(n + 1)^{\frac{n+1}{2}} .$$

524 & 525 Show that the result of problem 522 is exact in many cases: $n = 1, 2, 2^k, 12, \cdots$. Thus the result cannot be improved.

526 Show that of all n-th order matrices with real entries between -1 and +1 inclusive, there is at least one matrix of maximal determinant. Also show that this maximum is divisible by 2^{n-1}.

*527 Find the true upper bound for the absolute value of the determinant of a square matrix of real numbers of order 3 or 5, the elements of the matrix being between -1 and +1.

*528 Let $A = [a_{ij}]$. The obverse matrix has for ik-th element the minor of A obtained by deleting the i-th row in k-th column. Show that the obverse of the obverse matrix has for ik-th element $a_{ik} \cdot \Delta^{n-2}$, where $\Delta = \det A$.

*529 Show that the m-th order minors of the obverse matrix can be obtained from the corresponding minors of the original matrix by multiplication with the factor Δ^{m-1}.

530 Show that the obverse of the product of two matrices is equal to the product of the obverses in the same order.

531 Let the combinations of m sets of the integers $1, 2, \cdots, n$ be indexed in some manner, for instance in lexicographic order.

Let $A = (a_{ik})$ be an m-th order matrix. Let $A_{\alpha\beta}$ be an m-th order minor of A obtained by taking the rows of the α-th combination of rows and the columns of the β-th combination of columns. The matrix of all such minors $(A_{\alpha\beta})$ is called the m-th compound $A_{(m)}$, and has order C^n_m. In particular $A_1 = A$; A_{n-1} is the obverse of the matrix A.

Establish the following identities:

$$(AB)_{(m)} = A_{(m)}B_{(m)},\ E_{(m)} = E,\ (A^{-1})_{(m)} = (A_{(m)})^{-1}.$$

532 Let A be a triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

If the combinations of m subsets of the integers $1, 2, \cdots, n$ are suitably numbered, then the matrix $A_{(m)}$ is also triagonal.

533 Show that the determinant of the matrix $A_{(m)}$ is the C_{m-1}^{n-1} power of the determinant of A.

534 Let the set of all pairs (i, k); i = 1,2,···, n; k = 1,2,···, m be numbered in an arbitrary manner (for example, in lexicographic order). In particular let the number of the pair (i_1, k_1) be α_1 and let the number of the pair (i_2, k_2) be α_2. By the Kronecker product of the matrices A, B of orders n, m respectively, we mean the nm-th order matrix $C = A \times B$, with elements

$$c_{\alpha_1\alpha_2} = a_{i_1 i_2} b_{k_1 k_2}$$

Establish the following relations, where the symbols denote arbitrary matrices of suitable orders:

a) $(A_1 \pm A_2) \times B = (A_1 \times B) \pm (A_2 \times B)$,

b) $A \times (B_1 \pm B_2) = (A \times B_1) \pm (A \times B_2)$,

c) $(A' \times B') \cdot (A'' \times B'') = (A' \cdot A'') \times (B' \cdot B'')$.

*535 Show that the value of the determinant of the product $A \times B$ is $(\det A)^m (\det B)^n$.

536 Suppose the mn-th order matrices A, B are partitioned into n^2 boxes in the following way:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{pmatrix},$$

where A_{ik}, B_{ik} are all square matrices of order m. Let $C = A \cdot B$ be the product of A and B and suppose that C is partitioned conformally so that the boxes are denoted C_{ik}. Establish the relation

$$C_{ik} = A_{i1}B_{1k} + A_{i2}B_{2k} + \dots + A_{in}B_{nk}.$$

Thus, a proper way to multiply partitioned matrices is to treat the boxes formally as elements of a ring.

*537 Let C be a matrix of order mn, partitioned into n^2 square boxes each of dimension m. Let the sub-matrices so obtained be denoted by A_{ik} and suppose that every two of these sub-matrices are commutative. One can form the "determinant"

$$\sum \pm A_{1\alpha_1} A_{2\alpha_2} \dots A_{n\alpha_n} = B$$

by considering these sub-matrices as elements of a large matrix. This determinant is in fact a matrix of order m. Show that the determinant of the matrix C is equal to the determinant of the original matrix B.

CHAPTER V - PROBLEMS

POLYNOMIALS AND RATIONAL FUNCTIONS OF A SINGLE INDETERMINATE

1. PROPERTIES OF POLYNOMIALS. TAYLOR'S FORMULA. MULTIPLE ROOTS

538 Multiply the following products of polynomials:

a) $(2x^4 - x^3 + x^2 + x + 1)(x^2 - 3x + 1)$;
b) $(x^3 + x^2 - x - 1)(x^2 - 2x - 1)$.

539 Find the quotient and remainder for the following problems:

a) $2x^4 - 3x^3 + 4x^2 - 5x + 6 \div x^2 - 3x + 1$;
b) $x^3 - 3x^2 - x - 1 \div 3x^2 - 2x + 1$.

540 Find conditions on the coefficients so that $x^3 + px + q$ be exactly divisible by $x^2 + mx - 1$.

541 Find conditions on the coefficients so that the polynomial $x^4 + px^2 + q$ be exactly divisible by $x^2 + mx + 1$.

542 Simplify the following polynomial

$$1 - \frac{x}{1} + \frac{x(x-1)}{1 \cdot 2} - \dots + (-1)^n \frac{x(x-1)\dots(x-n+1)}{n!}.$$

543 In the following problems, find the quotient and remainder:

a) $x^4-2x^3+4x^2-6x+8 \div x-1$;
b) $2x^5-5x^3-8x \div x+3$;
c) $4x^3+x^2 \div x+1+i$;
d) $x^3-x^2-x \div x-1+2i$.

544 Use synthetic division to calculate $f(x_0)$:

a) $f(x)=x^4-3x^3+6x^2-10x+16,\ x_0=4$;
b) $f(x)=x^5+(1+2i)x^4-(1+3i)x^2+7,\ x_0=-2-i$.

545 Use synthetic division to expand the polynomial $f(x)$ in powers of $x - x_0$:

a) $f(x)=x^4+2x^3-3x^2-4x+1,\quad x_0=-1$;
b) $f(x)=x^5\quad x_0=1$;
c) $f(x)=x^4-8x^3+24x^2-50x+90,\quad x_0=2$;
d) $f(x)=x^4+2ix^3-(1+i)x^2-3x+7+i,\quad x_0=-i$;
e) $f(x)=x^4+(3-8i)x^3-(21+18i)x^2-(33-20i)x+7+18i,\quad x_0=-1+2i$.

546 Use synthetic division to expand each of the following into partial fractions:

a) $\dfrac{x^3-x+1}{(x-2)^5}$; b) $\dfrac{x^4-2x^2+3}{(x+1)^5}$.

*547 Use synthetic division to write each of the following in powers of x:

a) $f(x + 3)$, where $f(x) = x^4 - x^3 + 1$;

b) $(x - 2)^4 + 4(x - 2)^3 + 6(x - 2)^2 + 10(x - 2) + 20$.

548 For the given value $x = x_0$ find the value of the polynomial $f(x)$ and its derivative $f'(x)$.

a) $f(x) = x^5 - 4x^3 + 6x^2 - 8x + 10, \quad x_0 = 2$;
b) $f(x) = x^4 - 3ix^3 - 4x^2 + 5ix - 1; \quad x_0 = 1 + 2i$.

549 Find the multiplicity of each number as a root of the polynomial written beside it:

a) 2 for the polynomial $x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$;

b) -2 for the polynomial $x^5 + 7x^4 + 16x^3 + 8x^2 - 16x - 16$?

550 For what values of the coefficient a will the polynomial $x^5 - ax^2 - ax + 1$ have -1 as a root of multiplicity at least two?

551 Find a, b if the trinomial $ax^4 + bx^3 + 1$ is divisible by $(x - 1)^2$.

552 Find a, b if the trinomial $ax^{n+1} + bx^n + 1$ is divisible by $(x - 1)^2$.

*553 Show that 1 is a three-fold root of each of the following polynomials:

a) $x^{2n} - nx^{n+1} + nx^{n-1} - 1$;
b) $x^{2n+1} - (2n + 1)x^{n+1} + (2n + 1)x^n - 1$;
c) $(n - 2m)x^n - nx^{n-m} + nx^m - (n - 2m)$

554 Show that the following polynomial is divisible by $(x - 1)^5$ but not by $(x - 1)^6$:

$$x^{2n+1} - \frac{n(n+1)(2n+1)}{6}x^{n+2} + \frac{(n-1)(n+2)(2n+1)}{2}x^{n+1} - \frac{(n-1)(n+2)(2n+1)}{2}x^{n} + \frac{n(n+1)(2n+1)}{6}x^{n-1} - 1$$

*555 Show that necessary and sufficient conditions that the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$$

should be divisible by $(x - 1)^{k+1}$ are:

$$\begin{aligned} a_0 + a_1 + a_2 + \dots + a_n &= 0, \\ a_1 + 2a_2 + \dots + na_n &= 0, \\ a_1 + 4a_2 + \dots + n^2a_n &= 0, \\ \dots\dots\dots\dots\dots \\ a_1 + 2^ka_2 + \dots + n^ka_n &= 0. \end{aligned}$$

556 Let $f(x)$ be a polynomial. Find the multiplicity of a as a root of:

$$\frac{x-a}{2}[f'(x) + f'(a)] - f(x) + f(a),$$

557 Find conditions under which the polynomial $x^5 + ax^3 + b$ should have a double root different from 0.

558 Find conditions that the polynomial $x^5 + 10a\,x^3 + 5b\,x + c$ should have a triple root different from 0.

559 Show that the trinomial $x^n + ax^{n-m} + b$ cannot have a nonzero multiple root of multiplicity 3 or higher.

560 Find conditions under which the trinomial $x^n + ax^{n-m} + b$ should have a nonzero double root.

*561 Show that the k-nomial

$$a_1x^{p_1} + a_2x^{p_2} + \dots + a_kx^{p_k}$$

cannot have a nonzero multiple root of multiplicity exceeding $k - 1$.

*562 Show that every non-zero root of multiplicity $k - 1$ of the polynomial

$$a_1x^{p_1} + a_2x^{p_2} + \dots + a_kx^{p_k}$$

must satisfy the following conditions

$$a_1x^{p_1}\varphi'(p_1) = a_2x^{p_2}\varphi'(p_2) = \dots = a_kx^{p_k}\varphi'(p_k),$$

where

$$\varphi(t) = (t - p_1)(t - p_2)(t - p_3)\dots(t - p_k),$$

and conversely.

*563 Show that a polynomial divisible by its own derivative must have the form $a_0(x - x_0)^n$.

564 Show that the polynomial

$$1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\ldots+\frac{x^n}{n!}$$

cannot have multiple roots.

565 Show that a necessary and sufficient condition that x_0 should be a root of multiplicity k for the rational function, $f(x)=\frac{\varphi(x)}{w(x)}$, it being understood that the denominator $w(x)$ does not reduce to zero for $x = x_0$, is that the following set of relations be satisfied:

$$f(x_0)=f'(x_0)=\ldots=f^{(k-1)}(x_0)=0, \quad f^{(k)}(x_0)\neq 0.$$

566 Show that the rational function $f(x)=\frac{\varphi(x)}{w(x)}$ can be written in the following form

$$f(x)=f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{h!}(x-x_0)^n+ \\ +\frac{F(x)}{w(x)}(x-x_0)^{n+1},$$

where $f(x)$ is a suitable polynomial. Here it is assumed that $w(x_0) \neq 0$. (Taylor's theorem for rational functions.)

*567 Show that if x_0 is a root of multiplicity k of the polynomial $f_1(x)\ f_2'(x) - f_2(x)\ f_1'(x)$,
then x_0 is a root of multiplicity k + 1 of the polynomial $f_1(x)f_2(x_0) - f_2(x)f_1(x_0)$
provided the latter is not identically zero. The converse is also true.

*568 Show that if f(x) has no multiple roots, then the polynomial $[f'(x)]^2 - f(x)\ f''(x)$ cannot have a root of multiplicity greater than n - 1, n being the degree of f(x).

*569 Construct a particular polynomial f(x) of degree n such that the polynomial $[f'(x)]^2 - f(x)\ f''(x)$ has a root x_0 of multiplicity of n - 1, where x_0 is not a root of f(x).

2. PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA AND RELATED QUESTIONS

570 Find a number δ with a property that the polynomial $x^5 - 4x^3 + 2x$ has absolute value less than 0.1 for all values of x such that $|x| < \delta$.

571 Let $f(x) = x^4 - 3x^3 + 4x + 5$. Find a number δ such that the relation $|f(x) - f(2)| < 0.01$ holds for all values x that satisfy the inequality $|x - 2| < \delta$.

572 Find a value of M so that the relation

$$|x^4 - 4x^3 + 4x^2 + 2| > 100.$$

is valid for all values of x greater than M in absolute value: $|x| > M$.

573 Find a variety, or closed set, of values of x such that the relation $|f(x)| < |f(0)|$ holds:

a) $f(x) = x^5 - 3ix^3 + 4$; b) $f(x) = x^5 - 3x^3 + 4$.

574 Find a large set of values of x such that the relation $|f(x)| < |f(1)|$ holds for the following polynomials f(x):

a) $f(x) = x^4 - 4x^3 + 2$;
b) $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 5$;
c) $f(x) = x^4 - 4x + 5$.

575 Let f(z) be the polynomial

$$f(z) = (1+i)z^5 + (3-5i)z^4 - (9+5i)z^3 - \\ - 7(1-i)z^2 + 2(1+3i)z + 4 - i.$$

Supposing that the relation $z - i = a(1 - i)$, $0 < a < \frac{1}{2}$, holds, show that $|f(z)| < \sqrt{5}$.

576 Let f(z) be an arbitrary non-constant polynomial. Show that in every neighborhood of z_0 there is a number z_1 such that the relation $|f(z_1)| > |f(z_0)|$ is satisfied.

577 Prove D'Alembert's lemma for rational functions.

578 Show that the maximum of the absolute value of a rational function of an independent variable is actually obtained when the variable varies over a rectangular region.

579 It is clear that the fundamental theorem of algebra (on the existence of roots of a polynomial equation)

is untrue for rational functions. Indeed, the function $1/z$ is never zero. At what point does the proof of the fundamental theorem algebra fail when we attempt to apply the proof to rational functions?

*580 Let $f(x)$ be a polynomial or rational function. Show that if a is a k-th order root of $f(z) - f(a)$, and $f(a) \neq 0$, then every circumference $|z - a| = \rho$ contains $2k$ points satisfying the relation $|f(z)| = |f(a)|$ provided ρ is sufficiently small.

*581 Let $f(x)$ be a polynomial or rational function. Show that if a is a k-fold root of $f(z) - f(a)$, then every sufficiently small circumference $|z - a| = \rho$ contains $2k$ points for which $\mathrm{Re}(f(z)) = \mathrm{Re}(f(a))$, and $2k$ points for which $\mathrm{Im}(f(z)) = \mathrm{Im}(f(a))$.

3. FACTORING INTO LINEAR FACTORS. FACTORING INTO FACTORS THAT ARE IRREDUCIBLE IN THE FIELD OF REAL NUMBERS. FORMULAS CONNECTING COEFFICIENTS AND ROOTS.

582 Factor each of the following polynomials into linear factors.

a) $x^3 - 6x^2 + 11x - 6$; b) $x^4 + 4$; c) $x^4 + 4x^3 + 4x^2 + 1$;
d) $x^4 - 10x^2 + 1$.

*583 Factor each of the following polynomials into linear factors:

a) $\cos(n \arccos x)$;
b) $(x + \cos\theta + i\sin\theta)^n + (x + \cos\theta - i\sin\theta)^n$;
c) $x^m - C_2^{2m} x^{m-1} + C_4^{2m} x^{m-2} - \dots + (-1)^m C_{2m}^{2m}$.

584 Factor each of the following into irreducible factors with real coefficients:

a) $x^4 + 4$; b) $x^6 + 27$; c) $x^4 + 4x^3 + 4x^2 + 1$;
d) $x^{2n} - 2x^n + 2$; e) $x^4 - ax^2 + 1, \ -2 < a < 2$;
f) $x^{2n} + x^n + 1$.

585 In each case find a polynomial of lowest possible degree having the roots indicated:

a) 1 as a double root; 2, 3, 1 + i as simple roots;

b) -1 as a triple root; 3, 4 as simple roots;

c) i as a double root; -1 -i as a simple root.

(It is not required that the coefficients be real.)

586 Find that the polynomial of lowest possible degree having as roots all roots of unity, of all degrees up to and including n.

587 Find a polynomial of lowest possible degree with real coefficients having the following roots:

a) 1 as a double root; 2, 3, 1 + i as simple roots;

b) 2 - 3i as a triple root;

c) i as a double root; -1 -i as a simple root.

588 For each pair of polynomials, find the greatest common divisor:

a) $(x-1)^3(x+2)^2(x-3)(x-4)$; $(x-1)^2(x+2)(x+5)$;
b) $(x-1)(x^2-1)(x^3-1)(x^4-1)$;
$(x+1)(x^2+1)(x^3+1)(x^4+1)$;
c) $(x^3-1)(x^2-2x+1)$; $(x^2-1)^3$.

*589 Find the greatest common divisor of the polynomials:

$$x^m-1 \;;\; x^n-1.$$

590 Find the greatest common divisor of the polynomials:

$$x^m+a^m \;;\; x^n+a^n.$$

591 Find the greatest common divisor of the polynomial f(x) and its derivative in the following three cases:

a) $f(x)=(x-1)^3(x+1)^2(x-3)$;
b) $f(x)=(x-1)(x^2-1)(x^3-1)(x^4-1)$;
c) $f(x)=x^{m+n}-x^m-x^n+1$.

592 Let f(x) be a polynomial with no multiple roots. Let x_0 be a k-fold root of the equation

$$f\left(\frac{u(x)}{v(x)}\right)=0$$

Show that the equation

$$f\left(\frac{u'(x)}{v'(x)}\right)=0$$

has x_0 as a k-1-fold root. [Assume that the relations $v(x_0)\neq 0$, $v'(x_0)\neq 0$ hold.]

593 If m, n, p are positive integers, show that $x^{3m}+x^{3n+1}+x^{3p+2}$ is divisible by $x^2 + x + 1$.

594 When will $x^{3m}-x^{3n+1}+x^{3p+2}$ be divisible by $x^2 - x + 1$?

595 Under what conditions is $x^{3m}+x^{3n+1}+x^{3p+2}$ divisible by $x^4 + x^2 + 1$?

596 Under what conditions is $x^{2m}+x^m+1$ divisible by $x^2 + x + 1$?

597 Show that

$$x^{ka_1}+x^{ka_2+1}+\dots+x^{ka_k+k-1}$$

is divisible by $x^{k-1}+x^{k-2}+\dots+1$.

598 For what values of m will $(x + 1)^m - x^m - 1$ be divisible by $x^2 + x + 1$?

599 For what values of m will $(x + 1)^m + x^m + 1$ be divisible by $x^2 + x + 1$?

600 For what values of m will $(x + 1)^m - x^m - 1$ be divisible by $(x^2 + x + 1)^2$?

601 For what values of m will $(x + 1)^m + x^m + 1$ be divisible by $(x^2 + x + 1)^2$?

602 Can the polynomials $(x + 1)^m + x^m + 1$ and $(x + 1)^m - x^m - 1$ be divisible by $(x^2 + x + 1)^3$?

603 Evaluate the polynomial

$$1-\frac{x}{1}+\frac{x(x-1)}{1\cdot 2}-\ldots+(-1)^n\frac{x(x-1)\ldots(x-n+1)}{1\cdot 2\ldots n},$$

when x takes on the sequence of values $1, 2, \cdots, n$. (See problem 542.)

604 Let X_n be the cyclotomic polynomial. For what values of m is $X_n(x^m)$ divisible by $X_n(x)$?

Demonstrate the following theorems:

605 If $f(x^n)$ is divisible by $x - 1$, then it is also divisible by $x^n - 1$.

606 If $f(x^n)$ is divisible by $(x - a)^k$, then it is also divisible by $(x^n - a^n)^k$, $a \neq 0$.

607 If $F(x)=f_1(x^3)+xf_2(x^3)$ is divisible by $x^2 + x + 1$, then $f_1(x)$, $f_2(x)$ are divisible by $x - 1$.

*608 Let $f(x)$ be a polynomial with real coefficients such that the relation $f(x) \geq 0$ holds for all real values of x. Show that $f(x)=[\varphi_1(x)]^2+[\varphi_2(x)]^2$, where $\varphi_1(x)$ and $\varphi_2(x)$ are polynomials with real coefficients.

609 Suppose the polynomial $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$ has roots $x_1, \cdots, x_n$. What are the roots of the following polynomials:

a) $a_0x^n-a_1x^{n-1}+a_2x^{n-2}-\ldots+(-1)^na_n$;
b) $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$;

c) $f(a)+\frac{f'(a)}{1}x+\frac{f''(a)}{1\cdot 2}x^2+\ldots+\frac{f(n)(a)}{n!}x^n$;

d) $a_0x^n+a_1bx^{n-1}+a_2b^2x^{n-2}+\ldots+a_nb^n$?

610 Find a relation among the coefficients of the cubic equation $x^3 + px^2 + qx + r = 0$, if one root is the sum of the other two.

611 Check that one root of the equation $36x^3 - 12x^2 - 5x + 1 = 0$ is equal to the sum of the other two. Solve the equation.

612 Find a relation among the coefficients of the fourth degree equation $x^4 + ax^3 + bx^2 + cx + d = 0$ if the sum of two roots is equal to the sum of the other two.

613 Show that an equation that fulfills the condition of problem 612 can be transformed into the form $y^4 + qy^2 + r = 0$ by the substitution $x = y + \alpha$, if α is properly chosen.

614 What relation among the coefficients of the fourth degree equation $x^4 + ax^3 + bx^2 + cx + d = 0$ guarantees that the product of two roots will be equal to the product of the other two?

615 Show that if an equation satisfies the conditions of 614, it can be solved by first dividing by x^2 and then making the substitution $y = x + \frac{c}{ax}$ $(a \neq 0)$.

616 Solve the equations:

$$\begin{aligned}
&\text{a) } x^4-4x^3+5x^2-2x-6=0;\\
&\text{b) } x^4+2x^3+2x^2+10x+25=0;\\
&\text{c) } x^4+2x^3+3x^2+2x-3=0;\\
&\text{d) } x^4+x^3-10x^2-2x+4=0,
\end{aligned}$$

by using the tricks in problems 612-615.

617 Find the value of λ such that one of the roots of the equation $x^3 - 7x + \lambda = 0$ is double another.

618 Find a, b, c so that they are roots of the equation

$$x^3 - ax^2 + bx - c = 0.$$

619 Find a, b, c so that they are roots of the equation

$$x^3 + ax^2 + bx + c = 0.$$

620 The sum of two roots of the equation

$$2x^3 - x^2 - 7x + \lambda = 0$$

is 1. Find λ.

621 Let x_1, x_2, x_3 be roots of the equation $x^3 + px + q = 0$. Find the relation among the coefficients if the equation $x_3 = \dfrac{1}{x_1} + \dfrac{1}{x_2}$ is valid.

622 Find the sum of the squares of the roots of the polynomial

$$x^n + a_1x^{n-1} + \ldots + a_n.$$

*623 Knowing the values of a_1, a_2 and knowing that the roots of the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0,$$

are in arithmetic progression, solve the equation.

624 Are the roots of the following equations in arithmetic progressions?

a) $8x^3 - 12x^2 - 2x + 3 = 0;$
b) $2x^4 + 8x^3 + 7x^2 - 2x - 2 = 0$

625 A fourth degree parabola has equation

$$y = x^4 + ax^3 + bx^2 + cx + d.$$

Find a straight line that intercepts this curve in points M_1, M_2, M_3, M_4 so that the three line segments cut off by the curve are equal: $M_1M_2 = M_2M_3 = M_3M_4$. Give conditions for the solvability of this problem.

*626 Find an equation of the fourth degree with roots:

$$\alpha, \ \frac{1}{\alpha}, \ -\alpha, \ -\frac{1}{\alpha}.$$

*627 Find an equation of the sixth degree with roots:

$$\alpha, \ \frac{1}{\alpha}, \ 1-\alpha, \ \frac{1}{1-\alpha}, \ 1-\frac{1}{\alpha}, \ \frac{1}{1-\frac{1}{\alpha}}.$$

628 Let f(x) be the polynomial

$$f(x)=(x-x_1)(x-x_2)\ldots(x-x_n).$$

Find formulas for $f'(x_i)$, $f''(x_i)$ and show that

$$\frac{\partial f'(x_i)}{\partial x_i}=\frac{1}{2}f''(x_i).$$

629 Let f(x) be the polynomial of problem 628. Suppose that $f(x_1)=f''(x_1)=0$, but $f'(x_1)\neq 0$. Show that

$$\sum_{i=2}^{n}\frac{1}{x_1-x_i}=0.$$

630 Suppose that the roots of the polynomial $x^n+a_1x^{n-1}+\ldots+a_n$ are in arithmetic progression. Find the value of $f'(x_i)$.

4. EUCLIDEAN ALGORITHM

631 Find the greatest common divisor of each pair of polynomials:

a) $x^4+x^3-3x^2-4x-1$, x^3+x^2-x-1;
b) $x^5+x^4-x^3-2x-1$, $3x^4+2x^3+x^2+2x-2$;
c) $x^6-7x^4+8x^3-7x+7$, $3x^5-7x^3+3x^2-7$;
d) $x^5-2x^4+x^3+7x^2-12x+10$, $3x^4-6x^3+5x^2+2x-2$;
e) $x^6+2x^4-4x^3-3x^2+8x-5$, x^5+x^2-x+1;
f) $x^5+3x^4-12x^3-52x^2-52x-12$,
$x^4+3x^3-6x^2-22x-12$;

g) $x^5+x^4-x^3-3x^2-3x-1$, $x^4-2x^3-x^2-2x+1$;
h) x^4-10x^2+1, $x^4-4\sqrt{2}\,x^3+6x^2+4\sqrt{2}\,x+1$;
i) $x^4+7x^3+19x^2+23x+10$, $x^4+7x^3+18x^2+22x+12$;
j) x^4-4x^3+1 , x^3-3x^2+1;
k) $2x^6-5x^5-14x^4+36x^3+86x^2+12x-31$,
$2x^5-9x^4+2x^3+37x^2+10x-14$;
l) $3x^6-x^5-9x^4-14x^3-11x^2-3x-1$,
$3x^5+8x^4+9x^3+15x^2+10x+9$.

632 Use the Euclidean algorithm to find polynomials $M_1(x)$, $M_2(x)$ so as to satisfy the relation
$f_1(x)M_2(x)+f_2(x)M_1(x)=\delta(x)$, where $\delta(x)$ is the greatest common divisor of the polynomials $f_1(x)$, $f_2(x)$:

a) $f_1(x)=x^4+2x^3-x^2-4x-2$,
$f_2(x)=x^4+x^3-x^2-2x-2$;
b) $f_1(x)=x^5+3x^4+x^3+x^2+3x+1$,
$f_2(x)=x^4+2x^3+x+2$;
c) $f_1(x)=x^6-4x^5+11x^4-27x^3+37x^2-35x+35$,
$f_2(x)=x^5-3x^4+7x^3-20x^2+10x-25$;
d) $f_1(x)=3x^7+6x^6-3x^5+4x^4+14x^3-6x^2-4x+4$,
$f_2(x)=3x^6-3x^4+7x^3-6x+2$;
e) $f_1(x)=3x^5+5x^4-16x^3-6x^2-5x-6$,
$f_2(x)=3x^4-4x^3-x^2-x-2$;
f) $f_1(x)=4x^4-2x^3-16x^2+5x+9$,
$f_2(x)=2x^3-x^2-5x+4$.

633 Use the Euclidean algorithm to find polynomials $M_1(x)$, $M_2(x)$ so as to satisfy the relation $f_1(x)M_2(x)+f_2(x)M_1(x)=1$:

a) $f_1(x)=3x^3-2x^2+x+2$, $f_2(x)=x^2-x+1$;
b) $f_1(x)=x^4-x^3-4x^2+4x+1$, $f_2(x)=x^2-x-1$;
c) $f_1(x)=x^5-5x^4-2x^3+12x^2-2x+12$,
$f_2(x)=x^3-5x^2-3x+17$;
d) $f_1(x)=2x^4+3x^3-3x^2-5x+2$,
$f_2(x)=2x^3+x^2-x-1$;
e) $f_1(x)=3x^4-5x^3+4x^2-2x+1$,
$f_2(x)=3x^3-2x^2+x-1$;
f) $f_1(x)=x^5+5x^4+9x^3+7x^2+5x+3$,
$f_2(x)=x^4+2x^3+2x^2+x+1$.

634 Use the method of undetermined coefficients to find polynomials $M_1(x)$, $M_2(x)$ such that the relation $f_1(x)M_2(x)+f_2(x)M_1(x)=1$ holds:

a) $f_1(x)=x^4-4x^3+1$, $f_2(x)=x^3-3x^2+1$;
b) $f_1(x)=x^3$, $f_2(x)=(1-x)^2$;
c) $f_1(x)=x^4$, $f_2(x)=(1-x)^4$.

635 Find polynomials $M_1(x)$, $M_2(x)$ of lowest possible degree satisfying the following relations:

a) $(x^4-2x^3-4x^2+6x+1)M_1(x)+(x^3-5x-3)M_2(x)=x^4$;
b) $(x^4+2x^3+x+1)M_1(x)$
$+(x^4+x^3-2x^2+2x-1)M_2(x)=x^3-2x$.

636 Find a polynomial of lowest possible degree that gives remainder

a) $2x$ when divided by $(x - 1)^2$; $3x$ when divided by $(x - 2)^3$.

b) $x^2 + x + 1$ when divided by $x^4 - 2x^3 + 10x - 7$; $2x^2 - 3$ when divided by $x^4 - 2x^3 - 3x^2 + 13x - 10$.

*637 Find polynomials $M(x)$, $N(x)$ satisfying the relation

$$x^m M(x) + (1 - x)^n N(x) = 1.$$

638 Let $\delta(x)$ be the greatest common divisor of the polynomials $f_1(x)$, $f_2(x)$. Suppose that the relation $f_1(x) M(x) + f_2(x) N(x) = \delta(x)$ holds. What is the greatest common divisor of $M(x)$, $N(x)$?

639 Find the multiple factors of the following polynomials:

a) $x^6 - 6x^4 - 4x^3 + 9x^2 + 12x + 4$;
b) $x^5 - 10x^3 - 20x^2 - 15x - 4$;
c) $x^6 - 15x^4 + 8x^3 + 51x^2 - 72x + 27$;
d) $x^5 - 6x^4 + 16x^3 - 24x^2 + 20x - 8$;
e) $x^6 - 2x^5 - x^4 - 2x^3 + 5x^2 + 4x + 4$;
f) $x^7 - 3x^6 + 5x^5 - 7x^4 + 7x^3 - 5x^2 + 3x - 1$;
g) $x^8 + 2x^7 + 5x^6 + 6x^5 + 8x^4 + 6x^3 + 5x^2 + 2x + 1$.

5. THE INTERPOLATION PROBLEM; RATIONAL FUNCTIONS

640 Use Newton's method to find the polynomial of lowest possible degree that has values given by the tables:

a) $\dfrac{x \;|\; 0\;1\;2\;3\;4}{f(x)\;|\;1\;2\;3\;4\;6}$; b) $\dfrac{x \;|\; -1\;0\;1\;2\;3}{f(x)\;|\;6\;5\;0\;3\;2}$;

c) $\dfrac{x \;\Big|\; 1\;\frac{9}{4}\;4\;\frac{25}{4}}{f(x)\;\Big|\;1\;\frac{3}{2}\;2\;\frac{5}{2}}$, find $f(2)$; d) $\dfrac{x \;|\; 1\;2\;3\;\;4\;6}{f(x)\;|\;5\;6\;1\;-4\;10}$.

641 Use Lagrange's formula to find a polynomial with the following table of values:

a) $\dfrac{x\;|\;1\;2\;3\;4}{y\;|\;2\;1\;4\;3}$; b) $\dfrac{x\;|\;1\;\;i\;-1\;-i}{y\;|\;1\;\;2\;\;3\;\;4}$.

*642 Find a polynomial f(x) with the following table of values:

$$\frac{x\;|\;1\;\varepsilon_1\;\varepsilon_2\;\ldots\;\varepsilon_{n-1}}{f(x)\;|\;1\;2\;3\;\ldots\;n},\quad \varepsilon_k=\cos\frac{2\pi k}{n}+i\sin\frac{2\pi k}{n}.$$

643 A polynomial f(x) has degree not exceeding n - 1. The polynomial takes the values $y_1, y_2, \cdots, y_n$ when x runs through the n-th roots of 1. Find the value of f(0).

*644 Establish the following theorem. A necessary and sufficient condition that the relation

$$f(x_0) = \frac{1}{n}[f(x_1)+f(x_2)+\ldots+f(x_n)]$$

should be valid for every polynomial f(x) of degree not exceeding n - 1 is that the points $x_1, x_2, \cdots, x_n$ be equally spaced on a circumference of a circle with center x_0.

*645 Let $\varphi(x)$ have roots $x_1, x_2, \cdots, x_n$. If these are all distinct show that

$$\sum_{i=1}^{n} \frac{x_i^s}{\varphi'(x_i)} = 0 \quad \text{for} \quad 0 \leqslant s \leqslant n-2.$$

646 In the preceding problem calculate the sum

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\varphi'(x_i)}$$

647 Show that Lagrange's interpolation formula can be obtained by using the solution of the following system of equations:

$$\begin{aligned} a_0+a_1x_1+\ldots+a_{n-1}x_1^{n-1} &= y_1, \\ a_0+a_1x_2+\ldots+a_{n-1}x_2^{n-1} &= y_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_0+a_1x_n+\ldots+a_{n-1}x_n^{n-1} &= y_n. \end{aligned}$$

*648 Find the polynomial of lowest possible degree with the following table of values

$$\begin{array}{c|ccccc} x & 0 & 1 & 2 & \dots & n \\ \hline y & 1 & 2 & 4 & \dots & 2^n \end{array}.$$

*649 Find the polynomial of lowest possible degree with the following table of values

$$\begin{array}{c|ccccc} x & 0 & 1 & 2 & \dots & n \\ \hline y & 1 & a & a^2 & \dots & a^n \end{array}.$$

*650 Find a polynomial of degree 2n that leaves remainder 1 when divided by $x(x - 2)\cdots(x - 2n)$ and leaves remainder -1 when divided by $(x - 1)(x - 3)\cdots[x - (2n - 1)]$.

*651 Find the polynomial of lowest possible degree with the following table of values

$$\begin{array}{c|ccccc} x & 1 & 2 & 3 & \dots & n \\ \hline y & 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \end{array}.$$

*652 Find a polynomial of degree not exceeding $(n - 1)$ for which the relation $f(x) = \dfrac{1}{x - a}$ is valid for each of the values $x_1, x_2, \cdots, x_n$, $(x_i \neq a,$ $i = 1, 2, \cdots, n)$.

*653 Show that if a polynomial has degree $k \leq n$ and has integral values for $n + 1$ consecutive integral values of the argument, then the polynomial takes integral values for all integral values of its argument.

*654 Show that if a polynomial has degree n and takes integral values for $x = 0, 1, 4, 9, \cdots, n^2$, then the value of the polynomial will be an integer for every integral square value of its argument.

*655 Resolve each of the following into partial fractions of the first kind:

a) $\frac{x^2}{(x-1)(x+2)(x+3)}$; b) $\frac{1}{(x-1)(x-2)(x-3)(x-4)}$;
c) $\frac{3+x}{(x-1)(x^2+1)}$; d) $\frac{x^2}{x^4-1}$; e) $\frac{1}{x^3-1}$; f) $\frac{1}{x^4+4}$;
g) $\frac{1}{x^n-1}$; h) $\frac{1}{x^n+1}$; i) $\frac{n!}{x(x-1)(x-2)\,\ldots\,(x-n)}$;
j) $\frac{(2n)!}{x(x^2-1)(x^2-4)\ldots(x^2-n^2)}$; k) $\frac{1}{\cos(n \arccos x)}$.

*656 Resolve each of the following into partial fractions of the first or second kind with real coefficients:

a) $\frac{1}{x^3-1}$; b) $\frac{x^2}{x^4-16}$; c) $\frac{1}{x^4+4}$; d) $\frac{x^2}{x^6+27}$;
e) $\frac{x^m}{x^{2n+1}-1}$, $m < 2n+1$; f) $\frac{x^m}{x^{2n+1}+1}$, $m < 2n+1$;
g) $\frac{1}{x^{2n}-1}$; h) $\frac{x^{2m}}{x^{2n}+1}$, $m < n$; i) $\frac{1}{x(x^2+1)(x^2+4)\ldots(x^2+n^2)}$.

*657 Resolve each of the following into partial fractions of the first kind:

$$\text{a) } \frac{x}{(x^2-1)^2}; \quad \text{b) } \frac{1}{(x^2-1)^2}; \quad \text{c) } \frac{5x^2+6x-23}{(x-1)^3(x+1)^3(x-2)}; \quad \text{d) } \frac{1}{(x^n-1)^2};$$

$$\text{e) } \frac{1}{x^m(1-x)^n}; \quad \text{f) } \frac{1}{(x^2-a^2)^n}, \quad a \neq 0; \quad \text{g) } \frac{1}{(x^2+a^2)^n}; \quad \text{h) } \frac{g(x)}{[f(x)]^2},$$

where $f(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$ is a polynomial having no multiple roots, and $g(x)$ is a polynomial of degree not exceeding $2n$.

658 Resolve each of the following into partial fractions of the first or second kind with real coefficients:

$$\text{a) } \frac{x}{(x+1)(x^2+1)^2}; \quad \text{b) } \frac{2x-1}{x(x+1)^2(x^2+x+1)^2};$$

$$\text{c) } \frac{1}{(x^4-1)^2}; \quad \text{d) } \frac{1}{(x^{2n}-1)^2}.$$

659 Set $\varphi(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$.
Show how to express each of the following in terms of $\varphi(x)$, $\varphi'(x)$, $\varphi''(x)$, etc:

$$\text{a) } \sum \frac{1}{x-x_i}; \quad \text{b) } \sum \frac{x_i}{x-x_i}; \quad \text{c) } \sum \frac{1}{(x-x_i)^2}.$$

*660 If $x_1, x_2, \cdots$ are known to be roots of the polynomial $\varphi(x)$ find the value of the following sums:

a) $\frac{1}{2-x_1}+\frac{1}{2-x_2}+\frac{1}{2-x_3}, \quad \varphi(x)=x^3-3x-1;$

b) $\frac{1}{x_1^2-3x_1+2}+\frac{1}{x_2^2-3x_2+2}+\frac{1}{x_3^2-3x_3+2},$
$\varphi(x)=x^3+x^2-4x+1;$

c) $\frac{1}{x_1^2-2x_1+1}+\frac{1}{x_2^2-2x_2+1}+\frac{1}{x_3^2-2x_3+1}, \quad \varphi(x)=x^3+x^2-1.$

661 Find a first degree polynomial that takes values differing as little as possible (in the sense of least squares) from the values in the following table:

x	0	1	2	3	4
y	2.1	2.5	3.0	3.6	4.1

662 Find a second degree polynomial that takes values differing as little as possible (in the sense of least squares) from the values in the following table:

x	0	1	2	3	4
y	1	1.4	2	2.7	3.6

6. RATIONAL ROOTS OF POLYNOMIALS REDUCIBILITY AND IRREDUCIBILITY IN THE FIELD OF RATIONAL NUMBERS

663 Show that if p/q is a rational fraction in lowest terms, and is a root of polynomial

$$f(x)=a_0x^n+a_1x^{n-1}+\ \dots\ +a_n$$

the coefficients of which are integers, then:

1) q divides a_0;
2) p divides a_n;
3) p - mq divides f(m) if m is an integer. In particular p - q divides f(1); p + q divides f(-1).

664 Find the rational roots of the following polynomials:

a) $x^3-6x^2+15x-14$; b) $x^4-2x^3-8x^2+13x-24$;
c) $x^5-7x^3-12x^2+6x+36$;
d) $6x^4+19x^3-7x^2-26x+12$;
e) $24x^4-42x^3-77x^2+56x+60$;
f) $x^5-2x^4-4x^3+4x^2-5x+6$;
g) $24x^5+10x^4-x^3-19x^2-5x+6$;
h) $10x^4-13x^3+15x^2-18x-24$;
i) $x^4+2x^3-13x^2-38x-24$;

*665 Show that if $f(x)$ is a polynomial with integral coefficients, and if $f(0)$, $f(1)$ are both odd, then $f(x)$ has no integral roots.

*666 Let x_1, x_2 be integers. Suppose that a polynomial with rational coefficients takes values +1, -1 when the independent variable is x_1, x_2. Show that if the relation $|x_1 - x_2| > 2$ holds, then the polynomial can have no rational root. If this relation does not hold, then the only possible rational root is $\frac{1}{2}(x_1 + x_2)$.

*667 Use Eisenstein's criterion to show that each of the following polynomials is irreducible:

a) $x^4 - 8x^3 + 12x^2 - 6x + 2$;
b) $x^5 - 12x^3 + 36x - 12$; c) $x^4 - x^3 + 2x + 1$.

*668 Show that, if p is a prime, the polynomial

$$X_p(x) = \frac{x^p - 1}{x - 1}$$

is irreducible.

*669 The same as 668 for the polynomial

$$X_{p^k}(x) = \frac{x^{p^k} - 1}{x^{p^{k-1}} - 1}$$

*670 Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ be a polynomial with integral coefficients and no rational roots. Show that the polynomial is irreducible if there is a prime p such that a_0 is not divisible by p, a_2, a_3, $\cdots$, a_n are divisible by p, and a_n is not divisible by p^2.

*671 Let f(x) be a polynomial with integral coefficients; let p be a prime such that a_0 is not divisible by p, $a_{k+1}, a_{k+2}, \cdots, a_n$ are divisible by p, and a_n is not divisible by p^2. Show that if f(x) has an irreducible divisor, it has one of degree n - k at least.

672 Substitute various integers for x: factor the polynomial values; in that way factor the corresponding polynomial or show that it is irreducible;

a) $x^4 - 3x^2 + 1$; b) $x^4 + 5x^3 - 3x^2 - 5x + 1$;
c) $x^4 + 3x^3 - 2x^2 - 2x + 1$; d) $x^4 - x^3 - 3x^2 + 2x + 2$.

673 Show that a third degree polynomial is irreducible if it has no rational root.

674 Show that the fourth degree polynomial $x^4 + ax^3 + bx^2 + cx + d$ with integral coefficients is irreducible if it has no integral roots and is not divisible by a polynomial of the form

$$x^2 + \frac{cm - am^2}{d - m^2} x + m,$$

where m is a divisor of d. In applying this criterion one may ignore cases in which the coefficient of x is not integral. The special polynomials in problems 614-615 form an exception.

675 Show that the fifth degree polynomial $x^5 + ax^4 + bx^3 + cx^2 + dx + e$ with integral coefficients is irreducible if it has no integral roots and has no factor of the form

$$x^2 + \frac{am^3 - cm^2 - dn + be}{m^3 - n^2 + ae - dm}\, x + m$$

with real coefficients, where m is a divisor of e: $n = \frac{e}{m}$.

676 Use problems 674, 675 to factor each of the following insofar as it is possible:

a) $x^4 - 3x^3 + 2x^2 + 3x - 9$; b) $x^4 - 3x^3 + 2x^2 + 2x - 6$;
c) $x^4 + 4x^3 - 6x^2 - 23x - 12$;
d) $x^5 + x^4 - 4x^3 + 9x^2 - 6x + 6$.

Note: An application of modular arithmetic to the problem of finding polynomial factors is given in the article: Factorization of the General Polynomial, by D.B. Lloyd. American Math. Monthly 71, (October 1964), 863-870.

See also:

Tables of Irreducible Polynomials for the First Four Prime Moduli, by Randolph Church. Annals of Math. 36 (1935), 198-209.

677 Find necessary and sufficient conditions for the reducibility of the polynomial $x^4 + px^2 + q$ if its coefficients are rational.

678 Show that a necessary (but not sufficient) condition that a fourth degree polynomial without rational roots should be reducible is that Ferrari's cubic resolvent should have a rational root.

679 Show that if $a_1, a_2, \cdots, a_n$ are distinct integers, then the polynomial

$$f(x) = (x - a_1)(x - a_2)\cdots(x - a_n) - 1$$

is irreducible.

680 In problem 679 if the -1 is replaced by +1, the theorem remains true with the following two exceptions:

$$(x-a)(x-a-1)(x-a-2)(x-a-3)+1= \\ =[(x-a-1)(x-a-2)-1]^2.$$

$$(x-a)(x-a-2)+1=(x-a-1)^2.$$

*681 If an n-th degree polynomial with integral coefficients takes the values +1, -1 for more than 2m integral values of the argument ($n = 2m$ or $2m + 1$), then it is irreducible.

*682 Show that if the numbers $a_1, a_2, \cdots, a_n$ are distinct prime numbers, the polynomial

$$f(x)=(x-a_1)^2(x-a_2)^2\ \ldots\ (x-a_n)^2+1$$

is irreducible.

*683 Show that if a polynomial $f(x)$ with integral coefficients assumes the value +1 for more than three different integral values of x, it cannot assume the value of -1 for any integral value of x.

*684 Show that if $n \geq 12$ and if an n-th degree polynomial with integral coefficients takes the values +1, -1 for more than n/2 integral values of its argument, then the polynomial is irreducible.

*685 Suppose $a_1, a_2, \cdots, a_n$ are distinct whole numbers, $n \geq 7$, and $\varphi(x) = (x - a_1)(x - a_2)\cdots(x - a_n)$. Show that the polynomial $ax^2 + bx + 1$ is irreducible, a, b integral, then the polynomial $a[\varphi(x)]^2 + b\varphi(x) + 1$ is also irreducible.

7. BOUNDS FOR THE ROOTS OF POLYNOMIALS

686 Let $a_0x^n + a_1x^{n-1} + \ldots + a_n$ be a polynomial with real or complex coefficients. Show that no root exceeds any of the following quantities in modulus:

a) $1 + \max\left|\frac{a_k}{a_0}\right|$, $k = 1, 2, \ldots, n$;

b) $\rho + \max\left|\frac{a_k}{a_0\rho^{k-1}}\right|$,

$k = 1, 2, \cdots, n$; and ρ is an arbitrary positive number;

c) $2\max\sqrt[k]{\left|\frac{a_k}{a_0}\right|}$, $k = 1, 2, \ldots, n$;

d) $\left|\frac{a_1}{a_0}\right| + \max\sqrt[k-1]{\left|\frac{a_k}{a_1}\right|}$, $k = 1, 2, \ldots, n$.

687 Show that no root of the polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$ can exceed the unique positive root of the polynomial

$b_0x^n - b_1x^{n-1} - b_2x^{n-2} - \ldots - b_n$ in modulus,

provided that $0 < b_0 \leqslant |a_0|,\ b_1 \geqslant |a_1|,\ b_2 \geqslant |a_2|,\ \ldots,\ b_n \geqslant |a_n|$.

688 Show that if $a_r \neq 0$, the roots of the polynomial $f(x) = a_0x^n + a_rx^{n-r} + \cdots + a_n$ cannot exceed any of the following quantities in modulus:

a) $1 + \sqrt[r]{\max\left|\frac{a_k}{a_0}\right|}$, $\quad k = r, \ldots, n$;

b) $\rho + \sqrt[r]{\max\left|\frac{a_k}{a_0\rho^{k-r}}\right|}$,

$k = r, \cdots, n$; ρ is an arbitrary positive number;

c) $\sqrt[r]{\left|\frac{a_r}{a_0}\right|} + \max \sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}$, $\quad k = r, \ldots, n$.

689 Given any polynomial with real coefficients, construct an auxiliary polynomial according to the following rule. Alter the sign of every coefficient (except the coefficient of the highest power) that has the sign of the coefficient of the highest power. Show that no root of the original polynomial can exceed in modulus the unique non-negative root of the auxiliary polynomial.

Prove the following theorems:

690 Let the polynomial $a_0x^n + a_1x^{n-1} + \dots + a_n$ have real coefficients, $a_0 > 0$. No real root of this polynomial exceeds:

a) $1 + \sqrt[r]{\max\left|\frac{a_k}{a_0}\right|}$, where r is the index of the first negative coefficient, and the index k in the max operation ranges over the indices of the negative coefficients.

b) $\rho + \sqrt[r]{\max\left|\frac{a_k}{a_0\rho^{k-r}}\right|}$, where r is the index of the first negative coefficient, the index k ranges over the indices of the negative coefficients, and ρ is an arbitrary positive number.

c) $2\max\sqrt[k]{\frac{|a_k|}{a_0}}$, where the index k ranges over the indices of the negative coefficients of the polynomial.

d) $\sqrt[r]{\frac{|a_r|}{a_0}} + \max\sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}$, where r is the index of the first negative coefficient, and k ranges over the indices of the negative coefficients.

691 A polynomial with non-negative coefficients cannot have a positive root.

692 If the relations $f(a) > 0,\ f'(a) \geqslant 0,\ \ldots,\ f^{(n)}(a) \geqslant 0$ hold, every real root of the polynomial f(x) is less than or equal to a.

693 Find upper bounds for the real roots of the following polynomials:

a) $x^4 - 4x^3 + 7x^2 - 8x + 3$; b) $x^5 + 7x^3 - 3$;
c) $x^7 - 108x^5 - 445x^3 + 900x^2 + 801$;
d) $x^4 + 4x^3 - 8x^2 - 10x + 14$.

8. STURM'S THEOREM

694 Construct the Sturm polynomials and isolate the roots of the following polynomials:

a) $x^3 - 3x - 1$; b) $x^3 + x^2 - 2x - 1$;
c) $x^3 - 7x + 7$; d) $x^3 - x + 5$; e) $x^3 + 3x - 5$.

695 Construct the Sturm polynomials and isolate the roots of the following polynomials:

a) $x^4 - 12x^2 - 16x - 4$; b) $x^4 - x - 1$;
c) $2x^4 - 8x^3 + 8x^2 - 1$; d) $x^4 + x^2 - 1$;
e) $x^4 + 4x^3 - 12x + 9$.

696 Construct the Sturm polynomials and isolate the roots of the following polynomials:

a) $x^4-2x^3-4x^2+5x+5$; b) $x^4-2x^3+x^2-2x+1$;
c) $x^4-2x^3-3x^2+2x+1$; d) $x^4-x^3+x^2-x-1$;
e) $x^4-4x^3-4x^2+4x+1$.

697 Construct the Sturm sequence and isolate the roots of the following polynomials:

a) $x^4-2x^3-7x^2+8x+1$; b) x^4-4x^2+x+1;
c) $x^4-x^3-x^2-x+1$; d) $x^4-4x^3+8x^2-12x+8$;
e) x^4-x^3-2x+1.

698 Construct the Sturm sequence and isolate the roots of the following polynomials

a) x^4-6x^2-4x+2; b) $4x^4-12x^2+8x-1$;
c) $3x^4+12x^3+9x^2-1$; d) $x^4-x^3-4x^2+4x+1$;
e) $9x^4-126x^2-252x-140$.

699 Construct the Sturm sequence and isolate the roots of the following polynomials:

a) $2x^5-10x^3+10x-3$;
b) $x^6-3x^5-3x^4+11x^3-3x^2-3x+1$;
c) $x^5+x^4-4x^3-3x^2+3x+1$; d) $x^5-5x^3-10x^2+2$.

700 Construct the Sturm sequence, evaluate the Sturm functions for suitable positive values, and isolate the roots of the polynomials.

a) $x^4 + 4x^2 - 1$; b) $x^4 - 2x^3 + 3x^2 - 9x + 1$;
c) $x^4 - 2x^3 + 2x^2 - 6x + 1$; d) $x^5 + 5x^4 + 10x^2 - 5x - 3$.

701 Use Sturm's theorem to determine the number of real roots of the equation $x^3 + px + q = 0$, where p, q are real.

*702 Find the number of real roots of the equation $x^n + px + q = 0$, where p,q are real.

703 Find the number of real roots of the equation $x^5 - 5ax^3 + 5a^2x + 2b = 0$, where a, b are real.

704 Show that if a Sturm sequence contains polynomials of all degrees from 0 to n, then the number of changes of sign in the sequence of coefficients of the highest powers in the Sturm polynomials is equal to the number of pairs of conjugate complex roots of the original polynomials.

705 Suppose that the polynomials $f(x)$, $f_1(x)$, $f_2(x)$, $\cdots$, $f_k(x)$ satisfy the following hypotheses:

1) $f(x)f_1(x)$ changes sign from + to - when x passes through a root of $f(x)$;
2) two adjacent members of the series of polynomials never vanish simultaneously;

3) If $f_\lambda(x_0) = 0$, then $f_{\lambda-1}(x_0)$ and $f_{\lambda+1}(x_0)$ have opposite signs;

4) The sequence of polynomials does not change sign in the interval (a,b). Show that the number of roots of the polynomial $f(x)$ in the interval (a,b) is equal to the increase in the number of changes of sign in the values of the sequence of polynomials $f_1, \cdots, f_k$ as x changes from a to b.

706 Let x_0 be a real root of the polynomial $f'(x)$; let

$$f_1(x) = \frac{1}{x - x_0} f'(x);$$

let $f_2(x)$ be the negative of the remainder when $f(x)$ is divided by $f_1(x)$; let $f_3(x)$ be the negative of the remainder when $f_1(x)$ is divided by $f_2(x)$; etc. Suppose that $f(x)$ has no multiple root. Find the connection between the number of real roots of $f(x)$ and the number of changes of sign of the sequence of polynomials between $x = -\infty$, $x = x_0$, $x = +\infty$.

*707 For the Hermite polynomials

$$P_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n}$$

construct the Sturm sequence and find the number of real roots.

*708 For the Laguerre polynomials, find the number of real roots:

$$P_n(x) = (-1)^n e^x \frac{d^n e^{-x} x^n}{dx^n}.$$

Find the number of real roots of the following polynomials:

*709 $$E_n(x) = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \dots + \frac{x^n}{n!}.$$

*710 $$P_n(x) = (-1)^{n+1} x^{2n+2} e^{-\frac{1}{x}} \frac{d^{n+1}\left(e^{\frac{1}{x}}\right)}{dx^{n+1}}.$$

*711 $$P_n(x) = \frac{(-1)^n}{n!}(x^2+1)^{n+1} \frac{d^n}{dx^n}\left(\frac{1}{x^2+1}\right).$$

*712 $$P_n(x) = (-1)^n (x^2+1)^{n+\frac{1}{2}} \frac{d^n}{dx^n}\left(\frac{1}{\sqrt{x^2+1}}\right).$$

*713 Let f(x) be a third degree polynomial without multiple roots. Show that the polynomial $F(x) = 2f(x) f''(x) - [f'(x)]^2$ has exactly two real roots. Discuss the case when f(x) has a double or triple root.

714 Show that if all roots of the polynomial f(x) are real and distinct, every root of each of the polynomials in the Sturm sequence constructed by Euclid's algorithm will be real and simple.

9. MISCELLANEOUS METHODS FOR THE LOCATION OF ROOTS OF POLYNOMIALS

Prove the following theorems:

715 All roots of the Legendre polynomial

$$P_n(x) = \frac{d^n (x^2 - 1)^n}{dx^n}$$

are real and lie in the interval $(-1, +1)$.

716 If all roots of a polynomial $f(x)$ with real coefficients are real, then all roots of the polynomial $\lambda f(x) + f'(x)$ are real, where λ is an arbitrary real number.

*717 If all roots of the polynomial $f(x)$ with real coefficients are real, and all roots of the polynomial $g(x) = a_0 x^n + a_1 x^{n-1} + \cdots a_n$ with real coefficients are real, then all roots of the polynomial

$$F(x) = a_0 f(x) + a_1 f'(x) + \ldots + a_n f^{(n)}(x)$$

are also real.

*718 If all roots of the polynomial

$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ with real coefficients are real, then all roots of the polynomial

$$a_0 x^n + a_1 m x^{n-1} + a_2 m(m-1) x^{n-2} + \ldots + a_n m(m-1)\ldots(m-n+1)$$

are real, where m is an arbitrary positive integer.

*719 If all roots of the polynomial

$f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$ with real coefficients are real, then all roots of the polynomial

$$G(x) = a_0x^n + C_1^n a_1x^{n-1} + C_2^n a_2x^{n-2} + \cdots + a_n$$

are real.

720 Show that all roots of the following polynomial are real

$$x^n+\left(\frac{n}{1}\right)^2 x^{n-1}+\left(\frac{n(n-1)}{1\cdot 2}\right)^2 x^{n-2}+\ldots+1.$$

*721 Find the number of real roots of the following polynomial

$$nx^n-x^{n-1}-x^{n-2}-\ldots-1.$$

722 Find the number of real roots of the polynomial

$$x^{2n_1+1}+x^{2n_2+1}+\ldots+x^{2n_k+1}+a.$$

723 Assuming that a, b, c, A, B, C are real, find the number of real roots of the polynomial $(x - a)(x - b)(x - c) - A^2(x - a) - B^2(x - b) - C^2(x - c)$.

724 Suppose that $a_1, a_2, \cdots, a_n, A_1, A_2, \cdots, A_n, B$ are real. Show that the polynomial

$$\varphi(x)=\frac{A_1^2}{x-a_1}+\frac{A_2^2}{x-a_2}+\ldots+\frac{A_n^2}{x-a_n}+B$$

can have no imaginary roots.

Establish the following theorems:

725 If a polynomial $f(x)$ with real coefficients has roots that are real and distinct, then $[f'(x)]^2 - f(x)\,f''(x)$ can have no real root.

726 If all the roots and coefficients of the polynomials $f(x)$, $\varphi(x)$ are real, simple, and interlace (that is, there is a root of the second polynomial between every two roots of the first, and a root of the first polynomial between every two roots of the second), then all roots of the equation $\lambda f(x) + \mu\varphi(x) = 0$ are real. Here λ, μ are arbitrary real numbers.

*727 (Converse of 726) Suppose that, for every real λ, μ, all roots of the polynomial $F(x) = \lambda f(x) + \mu\varphi(x)$ are real. Then the roots of the polynomials $f(x)$, $\varphi(x)$ interlace.

*728 Let $f(x)$ be a polynomial with real coefficients and no multiple root. If all the roots of $f'(x)$ are real and distinct, then the number of real roots of polynomial $[f'(x)]^2 - f(x)f''(x)$ is equal to the number of imaginary roots of the polynomial $f(x)$.

*729 If $f_1(x)$, $f_2(x)$ have real coefficients and real roots that interlace, then the roots of their derivatives interlace.

*730 If $f(x)$ is a polynomial with real coefficients and real roots, then all roots of the polynomial

$F(x)=\gamma f(x)+(\lambda+x)f'(x)$ are real if λ is an arbitrary real number and γ satisfies one of the relations $\gamma>0$, $\gamma<-n$.

*731 Let the coefficients of the polynomials

$$f(x)=a_0+a_1x+\ldots+a_nx^n$$

$$\varphi(x)=b_0+b_1x+\ldots+b_kx^k$$

be real. Let f(x) have real roots and let $\varphi(x)$ have real roots all of which lie outside the interval (0,n); then all the roots of the polynomial

$$a_0\varphi(0)+a_1\varphi(1)x+a_2\varphi(2)x^2+\ldots+a_n\varphi(n)x^n$$

are real.

*732 Let $\gamma>n-1$ and let all coefficients be real. If the polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ has real roots, then all roots of the polynomial

$$a_0+a_1\gamma x+a_2\gamma(\gamma-1)x^2+\ldots+a_n\gamma(\gamma-1)\ldots(\gamma-n+1)x^n$$

are real.

*733 Suppose $\gamma>n-1,\ \alpha>0$ and all coefficients are real. If the roots of the polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ are real, then all roots of the polynomial

$$a_0+\frac{\gamma}{\alpha}a_1x+\frac{\gamma(\gamma-1)}{\alpha(\alpha+1)}a_2x^2+\ldots+\frac{\gamma(\gamma-1)\ldots(\gamma-n+1)}{\alpha(\alpha+1)\ldots(\alpha+n-1)}a_nx^n$$

are real.

*734 Let $0 < w \leqslant 1$. If all coefficients are real, and the polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ has real roots, then all roots of the polynomial

$$a_0 + a_1 w x + a_2 w^4 x^2 + \ldots + a_n w^{n^2} x^n$$

are real.

*735 Let the coefficients of the polynomial $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be real and suppose its roots all have the same sign. Then all roots of the polynomial

$$a_0 \cos\varphi + a_1 \cos(\varphi+\theta) x + a_2 \cos(\varphi+2\theta) x^2 + \ldots + a_n \cos(\varphi+n\theta) x_n$$

are real. Here φ, θ are arbitrary real quantities.

*736 Let the numbers $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_n$ be real. If all roots of the polynomial

$$(a_0 + ib_0) + (a_1 + ib_1) x + \ldots + (a_n + ib_n) x^n$$

lie in the upper half plane, then all roots of the polynomials

$$a_0 + a_1 x + \ldots + a_n x^n \quad , \quad b_0 + b_1 x + \ldots + b_n x^n$$

are real and interlace.

*737 Let $\varphi(x)$, $\psi(x)$ have real coefficients and suppose all the roots are real and interlace. Then the imaginary parts of the roots of $\varphi(x) + i\psi(x)$ have constant sign.

*738 If all roots of the polynomial f(x) lie in the upper half plane, then all roots of its derivative lie in the upper half plane.

*739 Suppose all roots of a polynomial lie in some half plane. Then the roots of its derivative lie in the same half plane.

*740 (Generalization of 739) If the roots of a polynomial lie inside an arbitrary convex contour, then the roots of its derivative lie in the same contour.

*741 Let f(x) be an n-th degree polynomial with real coefficients and real roots. Then all roots of the polynomial $[f(x)]^2 + k^2[f'(x)]^2$ have imaginary part not exceeding kn in absolute value.

742 Let the coefficients of a polynomial f(x) be real; and suppose all roots of the polynomials f(x) - a, f(x) - b are real. Then all roots of the polynomial f(x) - λ are real, where a < λ < b.

*743 A necessary and sufficient condition that a polynomial $x^n + a_1x^{n-1} + \ldots + a_n$ with real coefficients should have roots the real parts of which have constant sign is that the roots of the polynomials

$$x^n - a_2x^{n-2} + a_4x^{n-4} - \ldots$$

$$a_1x^{n-1} - a_3x^{n-3} + \ldots$$

be real and interlace.

*744 Let a, b, c be real. Give necessary and sufficient conditions that the real parts of all roots of the equation $x^3 + ax^2 + bx + c = 0$ be negative.

*745 Let a, b, c, d be real. Give necessary and sufficient conditions that the real parts of all roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ be negative.

*746 Let a, b, c be real. Give necessary and sufficient conditions that no root of the equation $x^3 + ax^2 + bx + c = 0$ exceed 1 in modulus.

*747 Show that the conditions $a_0 \geqslant a_1 \geqslant a_2 \geqslant \dots \geqslant a_n \geqslant 0$ are sufficient to guarantee that no root of the polynomial $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ exceed 1 in modulus.

10. APPROXIMATE CALCULATION OF THE ROOTS OF POLYNOMIALS

748 Calculate to within 0.0001 the root of $x^3 - 3x^2 - 13x - 7 = 0$ lying in the interval $(-1, 0)$.

749 Calculate the real root of the polynomial $x^3 - 2x - 5 = 0$ to within 0.000001.

750 Find the real roots of the following polynomials to within 0.0001:

a) $x^3 - 10x - 5 = 0$; b) $x^3 + 2x - 30 = 0$;
c) $x^3 - 3x^2 - 4x + 1 = 0$; d) $x^3 - 3x^2 - x + 2 = 0$.

751 Divide a hemisphere of radius 1 into two equal parts by a plane parallel to the base.

752 Find the positive root of the equation $x^3 - 5x - 3 = 0$ to within 0.0001.

753 Find to within 0.0001 the roots of the equations below lying on the intervals marked:

a) $x^4 + 3x^3 - 9x - 9 = 0, \quad (1, 2);$
b) $x^4 - 4x^3 + 4x^2 - 4 = 0, \quad (-1, 0);$
c) $x^4 + 3x^3 + 4x^2 + x - 3 = 0, \quad (0, 1);$
d) $x^4 - 10x^2 - 16x + 5 = 0, \quad (0, 1);$
e) $x^4 - x^3 - 9x^2 + 10x - 10 = 0, \quad (-4, -3);$
f) $x^4 - 6x^2 + 12x - 8 = 0, \quad (1, 2);$
g) $x^4 - 3x^2 + 4x - 3 = 0, \quad (-3, -2);$
h) $x^4 - x^3 - 7x^2 - 8x - 6 = 0, \quad (3, 4);$
i) $x^4 - 3x^3 + 3x^2 - 2 = 0, \quad (1, 2).$

754 Find to within 0.0001 all real roots of the following equations:

a) $x^4 + 3x^3 - 4x - 1 = 0;$ b) $x^4 + 3x^3 - x^2 - 3x + 1 = 0;$
c) $x^4 - 6x^3 + 13x^2 - 10x + 1 = 0;$ d) $x^4 - 8x^3 - 2x^2 + 16x - 3 = 0;$
e) $x^4 - 5x^3 + 9x^2 - 5x - 1 = 0;$ f) $x^4 - 2x^3 - 6x^2 + 4x + 4 = 0;$
g) $x^4 + 2x^3 + 3x^2 + 2x - 2 = 0;$ h) $x^4 + 4x^3 - 4x^2 - 16x - 8 = 0.$

CHAPTER VI - PROBLEMS

SYMMETRIC FUNCTIONS

1. SYMMETRIC FUNCTIONS IN TERMS OF THE ELEMENTARY SYMMETRIC FUNCTIONS. CALCULATION OF THE SYMMETRIC FUNCTIONS OF THE ROOTS OF AN ALGEBRAIC EQUATION.

755 Express each of the following in terms of the elementary symmetric functions:

a) $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$;
b) $x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$;
c) $x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_2^2x_3^2 - 2x_3^2x_1^2$;
d) $x_1^5x_2^2 + x_1^2x_2^5 + x_1^5x_3^2 + x_1^2x_3^5 + x_2^5x_3^2 + x_2^2x_3^5$;
e) $(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$;
f) $(x_1^2 + x_2^2)(x_1^2 + x_3^2)(x_2^2 + x_3^2)$;
g) $(2x_1 - x_2 - x_3)(2x_2 - x_1 - x_3)(2x_3 - x_1 - x_2)$;
h) $(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$.

756 Express each of the following in terms of the elementary symmetric functions:

a) $(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)$;
b) $(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)$;
c) $(x_1 + x_2 - x_3 - x_4)(x_1 - x_2 + x_3 - x_4)(x_1 - x_2 - x_3 + x_4)$.

757 Symmetrize each of the following polynomials; express the symmetric functions so obtained in terms of the elementary symmetric functions:

a) $x_1^2 + \ldots$;
b) $x_1^3 + \ldots$;
c) $x_1^2 x_2 x_3 + \ldots$;
d) $x_1^2 x_2^2 + \ldots$;
e) $x_1^3 x_2 + \ldots$;
f) $x_1^4 + \ldots$;
g) $x_1^2 x_2^2 x_3 + \ldots$;
h) $x_1^3 x_2 x_3 + \ldots$;
i) $x_1^3 x_2^2 + \ldots$;
j) $x_1^4 x_2 + \ldots$;
k) $x_1^5 + \ldots$;
l) $x_1^2 x_2^2 x_3 x_4 + \ldots$;
m) $x_1^2 x_2^2 x_3^2 + \ldots$;
n) $x_1^3 x_2 x_3 x_4 + \ldots$;
o) $x_1^3 x_2^2 x_3 + \ldots$;
p) $x_1^3 x_2^3 + \ldots$;
q) $x_1^4 x_2 x_3 + \ldots$;
r) $x_1^4 x_2^2 + \ldots$;
s) $x_1^5 x_2 + \ldots$;
t) $x_1^6 + \ldots$

758 Express each of the following in terms of the elementary symmetric functions:

a) $(-x_1 + x_2 + x_3 + \ldots + x_n)^2 + (x_1 - x_2 + x_3 + \ldots + x_n)^2 + (x_1 + x_2 - x_3 + \ldots + x_n)^2 + \ldots + (x_1 + x_2 + x_3 + \ldots - x_n)^2$;

b) $(-x_1 + x_2 + x_3 + \ldots + x_n)(x_1 - x_2 + x_3 + \ldots + x_n) \ldots (x_1 + x_2 + \ldots - x_n)$.

759 Express each of the following in terms of the elementary symmetric functions:

a) $\sum_{i>k}(x_i - x_k)^2$; b) $\sum_{i>k}(x_i + x_k)^3$;

c) $\sum_{i>k}(x_i - x_k)^4$; d) $\sum_{\substack{i>k \\ j\neq i;\ j\neq k}}(x_i + x_k - x_j)^2$.

760 Express the symmetric polynomial with leading term below in terms of the elementary symmetric functions:

$$x_1^2x_2^2 \ldots x_k^2 + \ldots$$

761 Express the following in terms of the elementary symmetric functions

$$\sum(a_1x_{i_1} + a_2x_{i_2} + \ldots + a_nx_{i_n})^2.$$

Here the sum is extended over all possible permutations

$i_1, i_2, \cdots, i_n$ of the numbers $1, 2, \cdots, n$.

762 Express the following in terms of elementary symmetric functions:

a) $\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}+\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}$; b) $\frac{(x_1-x_2)^2}{x_1+x_2}+\frac{(x_2-x_3)^2}{x_2+x_3}+\frac{(x_3-x_1)^2}{x_3+x_1}$;

c) $\left(\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}\right)\left(\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}\right)$.

763 Express the following in terms of elementary symmetric functions:

a) $\frac{x_1x_2}{x_3x_4}+\frac{x_1x_3}{x_2x_4}+\frac{x_1x_4}{x_2x_3}+\frac{x_2x_3}{x_1x_4}+\frac{x_2x_4}{x_1x_3}+\frac{x_3x_4}{x_1x_2}$;

b) $\frac{x_1+x_2}{x_3+x_4}+\frac{x_1+x_3}{x_2+x_4}+\frac{x_1+x_4}{x_2+x_3}+\frac{x_2+x_3}{x_1+x_4}+\frac{x_2+x_4}{x_1+x_3}+\frac{x_3+x_4}{x_1+x_2}$.

764 Express the following in terms of elementary symmetric functions:

a) $\sum\frac{1}{x_i}$; b) $\sum\frac{1}{x_i^2}$; c) $\sum\limits_{i\neq j}\frac{x_i}{x_j}$; d) $\sum\limits_{i\neq j}\frac{x_i^2}{x_j^2}$; e) $\sum\limits_{i\neq j}\frac{x_i^2}{x_j}$; f) $\sum\limits_{\substack{i\neq j\\ i\neq k\\ j>k}}\frac{x_jx_k}{x_i}$.

765 Calculate the sum of the squares of the roots of the equation

$$x^3 + 2x - 3 = 0.$$

766 Calculate $x_1^3x_2+x_1x_2^3+x_2^3x_3+x_2x_3^3+x_3^3x_1+x_3x_1^3$ if the numbers x_i are the roots of the equation $x^3 - x^2 - 4x + 1 = 0$.

767 Let the x_i be roots of the equation

$$x^4 + x^3 - 2x^2 - 3x + 1 = 0.$$

Calculate the value of the symmetric function

$$x_1^3x_2x_3+\cdots$$

768 Let x_1, x_2, x_3 be the roots of the equation $x^3 + px + q = 0$.

Calculate the following:

a) $\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}+\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}$;

b) $x_1^4x_2^2+x_1^2x_2^4+x_2^4x_3^2+x_2^2x_3^4+x_3^4x_1^2+x_3^2x_1^4$;

c) $(x_1^2 - x_2x_3)(x_2^2 - x_1x_3)(x_3^2 - x_2x_1)$;

d) $(x_1 + x_2)^4(x_1 + x_3)^4(x_2 + x_3)^4$;

e) $\dfrac{x_1^2}{(x_2+1)(x_3+1)} + \dfrac{x_2^2}{(x_1+1)(x_3+1)} + \dfrac{x_3^2}{(x_1+1)(x_2+1)}$;

f) $\dfrac{x_1^2}{(x_1+1)^2} + \dfrac{x_2^2}{(x_2+1)^2} + \dfrac{x_3^2}{(x_3+1)^2}$.

769 Find the relation among the coefficients of the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

if the square of one of its roots is equal to the sum of the squares of the other two.

770 Establish the following theorem. Necessary and sufficient conditions that all roots of the cubic equation $x^3 + ax^2 + bx + c = 0$ have negative real part are:

$$a > 0; \quad ab - c > 0; \quad c > 0.$$

771 Find the area and radius of the circumscribed circle of the triangle, the lengths of whose sides are the roots of the cubic equation

$$x^3 - ax^2 + bx - c = 0.$$

*772 Find a relation among the coefficients of an equation whose roots are equal to the sines of the angles of a triangle.

773 Calculate the value of the symmetric function suggested, when $x_1, \cdots$ are roots of the equation $f(x) = 0$ that is given:

a) $x_1^4 x_2 + \ldots,\quad f(x) = 3x^3 - 5x^2 + 1;$

b) $x_1^3 x_2^3 + \ldots,\quad f(x) = 3x^4 - 2x^3 + 2x^2 + x - 1;$

c) $(x_1^2 + x_1 x_2 + x_2^2)(x_2^2 + x_2 x_3 + x_3^2)(x_3^2 + x_3 x_1 + x_1^2),$

$$f(x) = 5x^3 - 6x^2 + 7x - 8.$$

774 Express the following symmetric functions in terms of the coefficients of the equation

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

assuming x_1, x_2, x_3 are its roots:

a) $a_0^4 (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2;$

b) $a_0^4 (x_1^2 - x_2 x_3)(x_2^2 - x_1 x_3)(x_3^2 - x_1 x_2);$

c) $\dfrac{(x_1 - x_2)^2}{x_1 x_2} + \dfrac{(x_1 - x_3)^2}{x_1 x_3} + \dfrac{(x_2 - x_3)^2}{x_2 x_3};$

d) $a_0^4 (x_1^2 + x_1 x_2 + x_2^2)(x_2^2 + x_2 x_3 + x_3^2)(x_3^2 + x_3 x_1 + x_1^2).$

775 Let $x_1, x_2, \cdots, x_n$ be the roots of the polynomial

$$x^n + a_1 x^{n-1} + \ldots + a_n.$$

Prove that any symmetric polynomial of $x_2, x_3, \cdots, x_n$ can be expressed as a polynomial in x_1.

776 Use the result of 775 to solve problems 755 e), 755 g), 774 b), 774 d).

777 Let f_k be the k-th elementary symmetric function of $x_1, x_2, \cdots, x_n$. Calculate

$$\sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i}$$

778 Suppose one knows how to express the symmetric function $F(x_1, x_2, \cdots, x_n)$ in terms of the elementary symmetric functions. Show how to express

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}$$

in terms of the elementary symmetric functions.

Establish the following theorems:

779 If $F(x_1, x_2, \cdots, x_n)$ is a symmetric function that satisfies the relation

$$F(x_1+a,\ x_2+a,\ \ldots,\ x_n+a)=F(x_1,\ x_2,\ \ldots,\ x_n),$$

and if $\Phi(f_1, f_2, \ldots, f_n)$ is the formula for expressing F in terms of the elementary ones, then the relation

$$n\frac{\partial \Phi}{\partial f_1}+(n-1)f_1\frac{\partial \Phi}{\partial f_2}+\ldots+f_{n-1}\frac{\partial \Phi}{\partial f_n}=0$$

holds, and conversely.

780 Any homogeneous second degree symmetric polynomial that satisfies the first relation in problem 779 can be written in the form

$$\alpha\sum_{i<k}(x_i-x_k)^2$$

for some constant value of α.

781 Find the general form of a third degree symmetric polynomial that satisfies the first relation in problem 779.

782 Use the result of problem 779 to express

$$\sum_{i<j<k}(x_i-x_j)^2(x_i-x_k)^2(x_j-x_k)^2$$

in terms of the elementary symmetric functions.

783 Consider the collection of symmetric polynomials $F(x_1, x_2, \cdots, x_n)$ that satisfy the condition

$$F(x_1, x_2, \ldots, x_n)=F(x_1+a, x_2+a, \ldots, x_n+a)$$

Show that there is a basic set of such polynomials $\varphi_2, \varphi_3, \cdots, \varphi_n$, $n - 1$ in number, such that all remaining polynomials of the set can be written in terms of the polynomials $\varphi_2, \varphi_3, \cdots, \varphi_n$.

784 (Refer to 783) Show how to express each of the following in terms of φ_2, φ_3:

a) $(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$;
b) $(x_1-x_2)^4+(x_1-x_3)^4+(x_2-x_3)^4$.

785 (Refer to 783) Show how to express each of the following in terms of $\varphi_2, \varphi_3, \varphi_4$:

a) $(x_1+x_2-x_3-x_4)(x_1-x_2+x_3-x_4)(x_1-x_2-x_3+x_4)$;
b) $(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2$.

2. POWER SUMS

786 Use Newton's formula to express s_2, s_3, s_4, s_5, s_6 in terms of the elementary symmetric polynomials.

787 Use Newton's formula to express the elementary symmetric functions f_2, f_3, f_4, f_5, f_6 in terms of the power sums $s_1, s_2, \cdots$.

788 Find the sum of the fifth powers of the roots of the equation

$$x^6 - 4x^5 + 3x^3 - 4x^2 + x + 1 = 0.$$

789 Find the sum of the eighth powers of the roots of the equation

$$x^4 - x^3 - 1 = 0.$$

790 Find the sum of the tenth powers of the roots of the equation

$$x^3 - 3x + 1 = 0.$$

791 Calculate $s_1, s_2, \cdots, s_n$ for the roots of the equation

$$x^n + \frac{x^{n-1}}{1} + \frac{x^{n-2}}{1\cdot 2} + \ldots + \frac{1}{n!} = 0.$$

792 Establish the formula

$$a^k(x_1^k + x_2^k) = (-1)^k\Big[b^k - \frac{k}{1} b^{k-2}ac + \frac{k(k-3)}{1\cdot 2} b^{k-4}a^2c^2 - \\ - \frac{k(k-4)(k-5)}{1\cdot 2\cdot 3} b^{k-6}a^3c^3 + \ldots\Big]$$

where x_1, x_2 are the roots of the quadratic equation $ax^2 + bx + c = 0$.

793 Let x_1, x_2, x_3 be the roots of an arbitrary third degree polynomial. Establish the following:

$$\frac{s_1^5 - s_5}{s_1^3 - s_3} = \frac{5}{3}(f_1^2 - f_2).$$

794 For a fourth degree polynomial, show that if the sum of the roots is 0, then:

$$\frac{s_5}{5} = \frac{s_3}{3} \cdot \frac{s_2}{2}.$$

795 For a sixth degree polynomial, show that if $s_1 = s_3 = 0$, then

$$\frac{s_7}{7} = \frac{s_5}{5} \cdot \frac{s_2}{2}.$$

796 Find an n-th degree polynomial for which

$$s_1 = s_2 = \dots = s_{n-1} = 0.$$

797 Find an n-th degree polynomial for which

$$s_2 = s_3 = \cdots = s_n = 0.$$

798 Find an n-th degree polynomial for which

$$s_2 = 1,\ s_3 = s_4 = \dots = s_n = s_{n+1} = 0.$$

799 Express $\sum_{i<j} x_i^k x_j^k$ in terms of the power sums.

*800 Express $\sum_{i<j} (x_i + x_j)^k$ in terms of the power sums.

*801 Express $\sum_{i<j} (x_i - x_j)^{2k}$ in terms of the power sums.

802 Show that

$$s_k = \det \begin{bmatrix} f_1 & 1 & & \dots & 0 \\ 2f_2 & f_1 & 1 & \dots & 0 \\ 3f_3 & f_2 & f_1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ kf_k & f_{k-1}, & f_{k-2} & \dots & f_1 \end{bmatrix}.$$

803 Show that

$$f_k = \frac{1}{k!} \det \begin{bmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ s_3 & s_2 & s_1 & 3 \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_k & s_{k-1} & s_{k-2} & \dots & s_1 \end{bmatrix}.$$

804 Calculate the value of the determinant

$$\det \begin{bmatrix} x^n & x^{n-1} & x^{n-2} & \dots & 1 \\ s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_n & s_{n-1} & s_{n-2} & \dots & n \end{bmatrix}.$$

*805 Calculate s_m for the roots of the cyclotomic equation $X_n(x) = 0$.

*806 Show that the only possible values of the functions f_2, f_3, f_4 of the roots of the cyclotomic equation $X_n(x) = 0$ are 0, +1, -1.

*807 Solve the following system of equations:

$$\begin{aligned} x_1 + x_2 + \dots + x_n &= a, \\ x_1^2 + x_2^2 + \dots + x_n^2 &= a, \\ \dots\dots\dots\dots \\ x_1^n + x_2^n + \dots + x_n^n &= a \end{aligned}$$

and calculate $x_1^{n+1} + x_2^{n+1} + \dots + x_n^{n+1}$.

*808 Calculate the power sums $s_1, s_2, \cdots, s_n$ for the roots of the equation

$$x^n + (a+b)x^{n-1} + (a^2+ab+b^2)x^{n-2} + \dots \\ \dots + (a^n + a^{n-1}b + \dots + b^n) = 0.$$

*809 Calculate the power sums $s_1, s_2, \cdots, s_n$ for the roots of the equation

$$x^n + (a+b)x^{n-1} + (a^2+b^2)x^{n-2} + \dots + (a^n+b^n) = 0.$$

3. TRANSFORMATION OF EQUATIONS

810 Let x_1, x_2, x_3 be roots of the equation $x^2 + ax^2 + bx + c = 0$. Find an equation having the following roots:

a) $x_1+x_2,\ x_2+x_3,\ x_3+x_1;$
b) $(x_1-x_2)^2,\ (x_2-x_3)^2,\ (x_3-x_1)^2;$
c) $x_1^2-x_2x_3,\ x_2^2-x_3x_1,\ x_3^2-x_1x_2;$
d) $(x_1-x_2)(x_1-x_3),\ (x_2-x_1)(x_2-x_3),\ (x_3-x_1)(x_3-x_2);$
e) $x_1^2,\ x_2^2,\ x_3^2$; f) $x_1^3,\ x_2^3,\ x_3^3$,

811 Let $\varepsilon=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, and let x_1, x_2, x_3 be the roots of the equation $x^3 + ax^2 + bx + c = 0$. Find the equation that has the following roots

$$(x_1+x_2\varepsilon+x_3\varepsilon^2)^3 \quad , \quad (x_1+x_2\varepsilon^2+x_3\varepsilon)^3$$

812 Let x_1, x_2, x_3 be the roots of the cubic equation $x^3 + ax^2 + bx + c = 0$. Find an equation of lowest possible degree one of whose roots is

$\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}$, if its coefficients can be expressed rationally in terms of the coefficients of the given equation.

813 Let x_1, x_2, x_3 be the roots of the cubic equation $x^3 + ax^2 + bx + c = 0$. Find an equation of lowest possible degree one of whose roots is x_1/x_2, if

the coefficients of the equation can be expressed rationally in terms of the coefficients of the given equation.

814 Let x_1, x_2, x_3, x_4 be roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$. Find the equation of lowest possible degree so that its coefficients can be expressed rationally in terms of a, b, c, d and that has the following root:

a) $x_1x_2 + x_3x_4$;

b) $(x_1 + x_2 - x_3 - x_4)^2$;

c) x_1x_2;

d) $x_1 + x_2$;

e) $(x_1 - x_2)^2$.

815 Use the answers for 814 a), 814 b) to express the roots of a fourth degree equation in terms of the roots of the resolvent cubic of problem 814 a).

816 Find a formula for solving the equation

$$x^4 - 6ax^2 + bx - 3a^2 = 0.$$

817 Let x_1, x_2, x_3, x_4, x_5 be roots of the equation $x^5 + ax + b = 0$. Find an equation one of whose roots is the following:

$$(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)(x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1)$$

4. RESULTANTS AND DISCRIMINANTS

*818 (Hermite's rule) Show that the resultant of the polynomials

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_n \quad , \quad \varphi(x) = b_0 x^m + \ldots + b_m$$

is the determinant of the matrix, the rows of which are the coefficients of the remainders obtained when $\varphi(x), \ x\varphi(x), \ \ldots, \ x^{n-1}\varphi(x)$ are divided by f(x); the coefficients of the remainders must be arranged according to ascending powers of x.

Hint: Note that the remainder $r_k(x)$ of the division of $x^{k-1}\varphi(x)$ by f(x) is equal to the remainder of the division of $xr_{k-1}(x)$ by f(x).

*819 (Bézout's rule) Show that the resultant of the polynomials

$$\psi_k(x) = (a_0 x^{k-1} + a_1 x^{k-2} + \ldots + a_{k-1}) \varphi(x) -$$
$$-(b_0 x^{k-1} + b_1 x^{k-2} + \ldots + b_{k-1}) f(x), \quad k = 1, \ldots, n$$

is equal to the determinant of the matrix, the rows of which are the coefficients of the following polynomials of degree ≤ (n - 1):

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$
$$\varphi(x) = b_0 x^n + b_1 x^{n-1} + \ldots + b_n$$

Note that

$$\psi_1 = a_0\varphi - b_0 f,$$
$$\psi_k = x\psi_{k-1} + a_{k-1}\varphi - b_{k-1} f.$$

*820 If $n > m$, show that the resultant of the polynomials

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n, \quad \varphi(x) = b_0x^m + b_1x^{m-1} + \ldots + b_m$$

is the determinant of the matrix whose rows are the coefficients of the polynomials $\chi_k(x)$ of degree $k \le (n - 1)$ defined by the following formulas:

$$\chi_k(x) = x^{k-1}\varphi(x) \quad \text{for} \quad 1 \leqslant k \leqslant n - m;$$
$$\chi_k(x) = (a_0x^{k-n+m-1} + a_1x^{k-n+m-2} + \ \ldots \ + a_{k-n+m-1})\, x^{n-m}\varphi(x) -$$
$$- (b_0x^{k-n+m-1} + b_1x^{k-u+m-2} + \ \ldots \ + b_{k-n+m-1})\, f(x)$$

It is understood that the polynomials χ_k are written in terms of ascending powers of x.

Note that

$$\chi_{n-m+1} = a_0x^{n-m}\varphi(x) - b_0f(x),$$

$$\chi_k = x\chi_{k-1} + a_{k-n+m-1}x^{n-m}\varphi(x) - b_{k-n+m-1}f(x)$$

for $k > n - m + 1$.

821 Find the resultants of the following pairs of polynomials:

a) $x^3 - 3x^2 + 2x + 1$, $2x^2 - x - 1$;
b) $2x^3 - 3x^2 + 2x + 1$, $x^2 + x + 3$;
c) $2x^3 - 3x^2 - x + 2$, $x^4 - 2x^2 - 3x + 4$;
d) $3x^3 + 2x^2 + x + 1$, $2x^3 + x^2 - x - 1$;
e) $2x^4 - x^3 + 3$, $3x^3 - x^2 + 4$;
f) $a_0x^2 + a_1x + a_2$, $b_0x^2 + b_1x + b_2$.

822 For what values of λ will the following pairs of polynomials have a common root:

a) $x^3 - \lambda x + 2$, $x^2 + \lambda x + 2$;
b) $x^3 - 2\lambda x + \lambda^3$, $x^2 + \lambda^2 - 2$;
c) $x^3 + \lambda x^2 - 9$, $x^3 + \lambda x - 3$?

823 Eliminate x from the following system of two equations:

a) $x^2 - xy + y^2 = 3,$ $x^2y + xy^2 = 6;$
b) $x^3 - xy - y^3 + y = 0,$ $x^2 + x - y^2 - 1 = 0;$
c) $y = x^3 - 2x^2 - 6x + 8,$ $y = 2x^3 - 8x^2 + 5x + 2.$

824 Solve each of the following five systems:

a) $y^2-7xy+4x^2+13x-2y-3=0,$
$y^2-14xy+9x^2+28x-4y-5=0;$

b) $y^2+x^2-y-3x=0,$
$y^2-6xy-x^2+11y+7x-12=0;$

c) $5y^2-6xy+5x^2-16=0,$
$y^2-xy+2x^2-y-x-4=0;$

d) $y^2+(x-4)y+x^2-2x+3=0,$
$y^3-5y^2+(x+7)y+x^3-x^2-5x-3=0;$

e) $2y^3-4xy^2-(2x^2-12x+8)y+x^3+6x^2-16x=0,$
$4y^3-(3x+10)y^2-(4x^2-24x+16)y-3x^3+$
$+2x^2-12x+40=0.$

825 Obtain the resultant of the following polynomials

$$a_0x^n+a_1x^{n-1}+\dots+a_n\,,\quad a_0x^{n-1}+a_1x^{n-2}+\dots+a_{n-1}.$$

826 Establish the relation

$$\Re(f,\ \varphi_1\cdot\varphi_2)=\Re(f,\ \varphi_1)\cdot\Re(f,\ \varphi_2).$$

*827 Find the resultant of the polynomials

$$X_n\ ,\ x^m-1.$$

*828 Find the resultant of the polynomials X_m, X_n (cyclotomic polynomials).

829 Calculate the discriminant of each of the five polynomials below:

a) $x^3-x^2-2x+1;$ b) $x^3+2x^2+4x+1;$
c) $3x^3+3x^2+5x+2;$ d) $x^4-x^3-3x^2+x+1;$
e) $2x^4-x^3-4x^2+x+1.$

830 Calculate the discriminant of each of the polynomials below:

a) $x^5 - 5ax^3 + 5a^2x - b$; b) $(x^2 - x + 1)^3 - \lambda(x^2 - x)^2$;
c) $ax^3 - bx^2 + (b - 3a)x + a$;
d) $x^4 - \lambda x^3 + 3(\lambda - 4)x^2 - 2(\lambda - 8)x - 4$.

831 For what values of λ will the following polynomials have multiple roots:

a) $\lambda^3 - 3x + \lambda$; b) $x^4 - 4x + \lambda$;
c) $x^3 - 8x^2 + (13 - \lambda)x - (6 + 2\lambda)$;
d) $x^4 - 4x^3 + (2 - \lambda)x^2 + 2x - 2$?

832 Show how the number of real roots of the polynomial with real coefficients is characterized by the sign of its discriminant in the following cases:

a) a third-degree polynomial;
b) a fourth-degree polynomial;
c) a general polynomial.

833 Calculate the discriminant of the polynomial $x^n + a$.

*834 Calculate the discriminant of the polynomial $x^n + px + q$.

*835 Calculate the discriminant of the polynomial

$$a_0x^{m+n}+a_1x^m+a_2.$$

836 Knowing the discriminant of the polynomial

$$a_0x^n+a_1x^{n-1}+\dots+a_n,$$

find the discriminant of the polynomial

$$a_nx^n+a_{n-1}x^{n-1}+\dots+a_0.$$

837 Show that the discriminant of a fourth-degree polynomial is equal to the discriminant of its Ferrari resolvent [see problems 814 a) and 80].

838 If D is the discriminant, show that

$$D((x-a)f(x))=D(f(x))[f(a)]^2.$$

*839 Calculate the discriminant of the polynomial

$$x^{n-1}+x^{n-2}+\dots+1.$$

*840 Calculate the discriminant of the polynomial

$$x^n+ax^{n-1}+ax^{n-2}+\dots+a.$$

841 Show that the discriminant of the product of two polynomials is equal to the product of the discriminants diminished by the square of their resolvent polynomial.

842 Find the discriminant of the polynomial

$$X_{p^m}=\frac{x^{p^m}-1}{x^{p^{m-1}}-1}.$$

*843 Find the discriminant of the cyclotomic polynomial X_n.

*844 Calculate the discriminant of the polynomial

$$E_n = n!\left(1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \ldots + \frac{x^n}{n!}\right).$$

*845 Calculate the discriminant of the polynomial

$$F_n = x^n + \frac{a}{1}x^{n-1} + \frac{a(a-1)}{1\cdot 2}x^{n-2} + \ldots + \frac{a(a-1)\ldots(a-n+1)}{n!}.$$

*846 Calculate the discriminant of the Hermite polynomial

$$P_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n}.$$

*847 Calculate the discriminant of the Laguere polynomial

$$P_n(x) = (-1)^n \frac{d^n(x^n e^{-x})}{dx^n}.$$

*848 Calculate the discriminant of the Čebyšev polynomial

$$2\cos\left(n \arccos\frac{x}{2}\right).$$

*849 Calculate the discriminant of the polynomial

$$P_n(x) = \frac{(-1)^n}{n!}(1+x^2)^{n+1}\frac{d^n\left(\frac{1}{1+x^2}\right)}{dx^n}.$$

*850 Calculate the discriminant of the polynomial

$$P_n(x) = (-1)^n (1+x^2)^{n+\frac{1}{2}} \frac{d^n \frac{1}{\sqrt{1+x^2}}}{dx^n}.$$

*851 Calculate the discriminant of the polynomial

$$P_n(x) = (-1)^n x^{2n+2} e^{-\frac{1}{x}} \frac{d^n \left(e^{\frac{1}{x}}\right)}{dx^n}.$$

*852 Calculate the maximum possible value of the discriminant of the polynomial

$$x^n + a_1 x^{n-1} + \dots + a_n,$$

if all its roots are real and satisfy the relation

$$x_1^2 + x_2^2 + \dots + x_n^2 = n(n-1)R^2.$$

853 Find the discriminant of $f(x^2)$ assuming the discriminant of $f(x)$ is known.

854 Find the discriminant of $f(x^m)$ assuming the discriminant of $f(x)$ is known.

855 Let $\varphi(x)$ be a polynomial of degree m; let $x_1, x_2, \cdots, x_n$ be the roots of the polynomial $f(x)$. Assume that the coefficient of the highest power of x in both f and φ is unity.

Show that the discriminant of $F(x) = f(\varphi(x))$ is given by the formula

$$[D(f)]^m \prod_{i=1}^{n} D(\varphi(x) - x_i)$$

5. TSCHIRNHAUS TRANSFORMATION AND THE ELIMINATION OF IRRATIONALITIES IN THE DENOMINATOR.

856 Use the substitution $y = x^2 - x - 1$ to transform the equation $(x - 1)(x - 3)(x + 4) = 0$.

857 Use the given substitutions to transform the given equations:

a) $x^3 - 3x - 4 = 0$ $\qquad y = x^2 + x + 1$;
b) $x^3 + 2x^2 + 2 = 0$ $\qquad y = x^2 + 1$;
c) $x^4 - x - 2 = 0$ $\qquad y = x^3 - 2$;
d) $x^4 - x^3 - x^2 + 1 = 0$ $\qquad y = x^3 + x^2 + x + 1$.

858 Make the given Tschirnhaus transformation in the given equations, and find the inverse transformation:

a) $x^3 - x + 2 = 0,$ $\qquad y = x^2 + x$;
b) $x^4 - 3x + 1 = 0,$ $\qquad y = x^3 + x$;
c) $x^4 + 5x^3 + 6x^2 - 1 = 0,$ $\qquad y = x^3 + 4x^2 + 3x - 1$.

859 Use the substitution $y = 2 - x^2$ to transform the equation $x^3 - x^2 - 2x + 1 = 0$ and give an interpretation of the final result.

860 Show that a necessary and sufficient condition that each root of a cubic equation with rational coefficients should be expressible rationally (with rational coefficients) in terms of any other

roots is that the discriminant should be the square of a rational number.

861 Express with no irrationalities in the denominator:

a) $\dfrac{1}{1+\sqrt{2}-\sqrt{3}}$; b) $\dfrac{1}{1+\sqrt[3]{2}+2\sqrt[3]{4}}$; c) $\dfrac{7}{1-\sqrt[4]{2}+\sqrt{2}}$.

862 If α satisfies the equation given write the corresponding quantity with rational denominator:

a) $\dfrac{\alpha}{\alpha+1}$, $\quad\alpha^3-3\alpha+1=0;$

b) $\dfrac{\alpha^2-3\alpha-1}{\alpha^2+2\alpha+1}$, $\quad\alpha^3+\alpha^2+3\alpha+4=0;$

c) $\dfrac{1}{3\alpha^3+\alpha^2-2\alpha-1}$, $\quad\alpha^4-\alpha^3+2\alpha+1=0,$

d) $\dfrac{1}{\alpha^3+3\alpha^2+3\alpha+2}$, $\quad\alpha^4+\alpha^3-4\alpha^2-3\alpha+2=0.$

863 Let x_1 be a root of the cubic equation $x^3 + ax^2 + bx + c = 0$. Show that every rational function of x_1 can be written in the form

$\dfrac{Ax_1+B}{Cx_1+D}$, where A, B, C, D are expressible rationally in terms of: x_1, a, b, c and the coefficients of the original rational function.

864 Let a cubic equation have rational coefficients, and be irreducible over the field of rational numbers; suppose its discriminant is rational. What sort of relation must subsist among the coefficients $\alpha, \beta, \gamma, \delta$, if the roots x_1, x_2 are connected by the rational relation

$$y_1,\ y_2,\ x_2 = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta}.$$

865 Carry out the transformation $y = x^2$ in the equation

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0.$$

866 Carry out the transformation $y = x^3$ in the equation

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0.$$

*867 Let the polynomial

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_n, \quad a_n \neq 0$$

have integral coefficients and suppose all its roots x_i satisfy the condition $|x_i| \leq 1$. Show that it is necessarily true that all roots are roots of unity.

6. POLYNOMIALS THAT ARE INVARIANT UNDER EVEN PERMUTATION OF THE VARIABLES. POLYNOMIALS THAT ARE INVARIANT UNDER CIRCULAR PERMUTATION OF THE VARIABLES.

868 Let a polynomial be invariant under even permutations of the variables and change sign under odd

permutations of the variables. Show that the polynomial is divisible by the Vandermondian (Vandermonde determinant in the independent variables) and the quotient of the division is a symmetric polynomial.

869 Show that if a polynomial does not change under even permutation of the variables, then it can be written in the form

$$F_1 + F_2 \Delta ,$$

where F_1, F_2 are symmetric polynomials and Δ is the Vandermondian.

870 Calculate

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^{n+1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^{n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^{n+1} \end{bmatrix}.$$

871 Let x_1, x_2, x_3 be roots of the equation $x^3 + ax^2 + bx + c = 0$. Find an equation having

$$\alpha x_1 + \beta x_2 + \gamma x_3, \ \alpha x_2 + \beta x_3 + \gamma x_1 \ , \ \alpha x_3 + \beta x_1 + \gamma x_2$$

as roots.

872 Let x_1, x_2, x_3 be roots of the equation $x^3 + px + q = 0$; y_1, y_2, y_3 roots of the equation $y_3 + p'y + q' = 0$. Find an equation having the quantities $x_1y_1 + x_2y_2 + x_3y_3$, $x_1y_2 + x_2y_3 + x_3y_1$, $x_1y_3 + x_2y_1 + x_3y_2$ as roots.

873 Prove that the following complicated condition is a necessary and sufficient one that the equations

$$x^3 + px + q = 0,$$
$$y^3 + p'y + q' = 0$$

with discriminants Δ, Δ' should be linked by a Tschirnhaus transformation. One of the numbers Δ, Δ' is the square of a rational number, and one of the equations

$$u^3 = 3pp'u + \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2}$$

should have rational roots.

874 Let a polynomial in the n indeterminates $x_1, x_2, \cdots, x_n$ be invariant under circular permutations of the variables. Show that the polynomial can be expressed in the form

$$\sum A f_1^{\alpha_0} \eta_1^{\alpha_1} \eta_2^{\alpha_2} \ldots \eta_{n-1}^{\alpha_{n-1}},$$

where $\eta_1, \eta_2, \ldots, \eta_{n-1}$ are the linear forms:

$$\begin{aligned} \eta_1 &= x_1\varepsilon + x_2\varepsilon^2 + \ldots + x_n, \\ \eta_2 &= x_1\varepsilon^2 + x_2\varepsilon^4 + \ldots + x_n, \\ &\cdots\cdots\cdots\cdots \\ \eta_{n-1} &= x_1\varepsilon^{n-1} + x_2\varepsilon^{2n-2} + \ldots + x_n; \quad \varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \end{aligned}$$

and where the individual terms in the sums satisfy the condition that the sum $\alpha_1 + 2\alpha_2 + \ldots + (n-1)\alpha_{n-1}$ is divisible by n.

875 Find a set of n fundamental rational functions (fractions with rational coefficients) in terms of which all rational functions that are invariant under circular permutations of the variables can be rationally expressed.

876 Show that if the number of variables is three, the fundamental set of problem 875 can be taken as a set of three functions with rational coefficients.

877 When the number of variables is four, find a set of four fundamental functions with rational coefficients, as in problem 876.

878 When the number of variables is five, find a set of five fundamental functions with rational coefficients, as in problem 877.

CHAPTER VII - PROBLEMS

LINEAR ALGEBRA

We use the following notation in this chapter. A <u>space</u> means a vector space over the field of rational numbers unless otherwise specified. The space may be one originally defined or may be a part of a more general space. In the latter case it is usually called a <u>subspace</u>. A <u>linear</u> <u>manifold</u> is the collection of all vectors $X_0 + X$, where X_0 is some fixed vector and X runs through the set of all vectors lying in some subspace.

The notation $X = (x_1, x_2, \cdots, x_n)$ means that the vector X has the numbers $x_1, x_2, \cdots, x_n$ as coordinates, when some fixed set of vectors is used as the basis for the space in question. When the space is the usual Euclidean one, the basis will be taken to be orthonormal.

A vector is called a point; a one-dimensional manifold a line; a two-dimensional manifold a plane.

1. SUBSPACES AND LINEAR VARIETIES. COORDINATE SUBSPACES.

879 A vector space is spanned by the vectors given in each of the three problems below. Find a basis, and the dimension of the space.

a) $X_1 = (2, 1, 3, 1)$, $X_2 = (1, 2, 0, 1)$, $X_3 = (-1, 1, -3, 0)$;

b) $X_1=(2,\ 0,\ 1,\ 3,\ -1)$ $\qquad X_2=(1,\ 1,\ 0,\ -1,\ 1)$,
$X_3=(0,\ -2,\ 1,\ 5,\ -3)$, $\qquad X_4=(1,\ -3,\ 2,\ 9,\ -5)$;

c) $X_1=(2,\ 1,\ 3,\ -1)$, $\qquad X_2=(-1\ ,1,\ -3,\ 1)$,
$X_3=(4,\ 5,\ 3,\ -1)$, $\qquad X_4=(1,\ 5,\ -3,\ 1)$.

880 In each of the three problems two spaces are given spanned respectively by the vectors X and the vectors Y. Form the sum and intersection of the spaces; find the dimensions of each; give a basis for each.

a) $X_1=(1,\ 2,\ 1,\ 0)$ $\qquad Y_1=(2,\ -1,\ 0,\ 1)$,
$X_2=(-1,\ 1,\ 1,\ 1)$, $\qquad Y_2=(1,\ -1,\ 3,\ 7)$;

b) $X_1=(1,\ 2,\ -1,\ -2)$, $\qquad Y_1=(2,\ 5,\ -6,\ -5)$,
$X_2=(3,\ 1,\ 1,\ 1)$, $\qquad Y_2=(-1,\ 2,\ -7,\ -3)$,
$X_3=(-1,\ 0,\ 1,\ -1)$;

c) $X_1=(1,\ 1,\ 0,\ 0)$, $\qquad Y_1=(0,\ 0,\ 1,\ 1)$,
$X_2=(1,\ 0,\ 1,\ 1)$, $\qquad Y_2=(0,\ 1,\ 1,\ 0)$.

881 Find the coordinates of the vector **X** if the basis is $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{E}_4$:

a) $X=(1, 2, 1, 1)$, $E_1=(1, 1, 1, 1)$, $E_2=(1, 1, -1, -1)$, $E_3=(1, -1, 1, -1)$, $E_4=(1, -1, -1, 1)$:
b) $X=(0, 0, 0, 1)$, $E_1=(1, 1, 0, 1)$, $E_2=(2, 1, 3, 1)$, $E_3=(1, 1, 0, 0)$, $E_4=(0, 1, -1, -1)$.

882 Find the matrix of coefficients and the formulas for the transformation of coordinates from the basis $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{E}_4$ to the basis $\mathbf{E}_1'$, $\mathbf{E}_2'$, $\mathbf{E}_3'$, $\mathbf{E}_4'$:

a) $E_1=(1, 0, 0, 0)$, $E_2=(0, 1, 0, 0)$, $E_3=(0, 0, 1, 0)$, $E_4=(0, 0, 0, 1)$, $E_1'=(1, 1, 0, 0)$, $E_2'=(1, 0, 1, 0)$, $E_3'=(1, 0, 0, 1)$, $E_4'=(1, 1, 1, 1,)$;
b) $E_1=(1, 2, -1, 0)$, $E_2=(1, -1, 1, 1)$, $E_3=(-1, 2, 1, 1)$, $E_4=(-1, -1, 0, 1)$, $E_1'=(2, 1, 0, 1)$, $E_2'=(0, 1, 2, 2)$, $E_3'=(-2, 1, 1, 2)$, $E_4'=(1, 3, 1, 2)$.

883 A certain surface has the equation $x_1^2+x_2^2-x_3^2-x_4^2=1$, in terms of the basis $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{E}_4$. Find the equation of the same surface with respect to the basis below, where the vectors are described by giving their coordinates in terms of the original basis:

$$E_1'=(1, 1, 1, 1);\quad E_2'=(1, 1, -1, -1);\quad E_3'=(1, -1, 1, -1);\quad E_4'=(1, -1, -1, 1).$$

*884 This question concerns the space of polynomials in cos x of degree $\leq n$. Give the formula for transformation of coordinates from the basis $1, \cos x, \cdots, \cos^n x$ to the basis $1, \cos x, \cdots, \cos nx$, and the inverse transformation.

885 In four-dimensional space, find the equation of the line that passes through the origin and intersects each of the lines:

$$x_1 = 2 + 3t, \quad x_2 = 1 - t, \quad x_3 = -1 + 2t, \quad x_4 = 3 - 2t$$
$$x_1 = 7t, \quad x_2 = 1, \quad x_3 = 1 + t, \quad x_4 = -1 + 2t.$$

Find the respective points of intersection.

886 Show that every two lines in n-dimensional space can be embedded in a three-dimensional subspace.

887 (See problem 885) Find conditions on two lines, given in an n-dimensional space, so that a third line can be found that intersects each of the two given lines, and passes through the origin.

888 In n-dimensional space, show that every two planes can be embedded in a five-dimensional subspace.

889 Describe all possible ways that two planes can be arranged in n-dimensional space (see the preceding problem).

890 Show that a linear manifold can be characterized as the set of all vectors containing with every two vectors X_1, X_2, all linear combinations of the form $\alpha X_1 + (1 - \alpha) X_2$.

2. ELEMENTARY GEOMETRY OF n-DIMENSIONAL EUCLIDEAN SPACE.

891 Find the scalar product of the vectors X, Y:

a) $X = (2, 1, -1, 2)$, $Y = (3, -1, -2, 1)$;

b) $X = (1, 2, 1, -1)$, $Y = (-2, 3, -5, -1)$.

892 Find the angle between the vectors X, Y:

a) $X = (2, 1, 3, 2)$, $Y = (1, 2, -2, 1)$;

b) $X = (1, 2, 2, 3)$, $Y = (3, 1, 5, 1)$;

c) $X = (1, 1, 1, 2)$, $Y = (3, 1, -1, 0)$.

893 Find the cosines of the angles between the line $x_1 = x_2 = \cdots = x_n$ and the coordinate axes.

894 Find the cosines of the interior angles of the triangle having as vertices:

$A = (1, 2, 1, 2)$, $B = (3, 1, -1, 0)$, $C = (1, 1, 0, 1)$.

895 Find the length of the diagonal of an n-dimensional cube that has edges of length 1.

896 Find the number of diagonals of an n-dimensional cube that are orthogonal to a given diagonal.

897 In n-dimensional space, find n points with non-negative coordinates arranged so that every one is at distance 1 from the origin and every two are

one unit apart. Take the first point on the first coordinate axis, the second in the plane determined by the first two axes and so on. (These points, together with the origin, lie in the vertices of a regular simplex of edge-length 1.)

898 Find the coordinates of the center and the radius of the circumscribed sphere of the simplex defined in problem 897.

899 Normalize the vector (3, 1, 2, 1).

900 Find a normalized vector orthogonal to each of the vectors (1, 1, 1, 1); (1, -1, -1, 1); (2, 1, 1, 3).

901 The two vectors below are given. Supplement them by two others and normalize so that an orthonormal basis of the space is obtained.

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad , \quad \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, -\frac{5}{6}\right)$$

902 Use the method of successive orthogonalization to find an orthogonal basis for the space spanned by the vectors (1, 2, 1, 3); (4, 1, 1, 1); (3, 1, 1, 0).

903 Find two row vectors that are mutually orthogonal and orthogonal to each of the three rows of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & -2 \\ 2 & 1 & -1 & 0 & 2 \end{pmatrix}$$

904 Give a geometric interpretation of the following system of homogeneous linear equations, and of a fundamental system of solutions of these equations in the n-dimensional space, the points of which have as coordinates the coefficient sets of the equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

905 Find a fundamental orthonormal system of solutions for the system of equations:

$$\begin{aligned} 3x_1 - x_2 - x_3 + x_4 &= 0, \\ x_1 + 2x_2 - x_3 - x_4 &= 0. \end{aligned}$$

906 In each of the two problems below, resolve the vector **X** into a sum of two vectors, one of which lies in the space spanned by the vectors A_1, A_2 (A_1, A_2, A_3) and the other of which is orthogonal to this space.

a) $X = (5, 2, -2, 2)$, $A_1 = (2, 1, 1, -1)$, $A_2 = (1, 1, 3, 0)$;
b) $X = (-3, 5, 9, 3)$, $A_1 = (1, 1, 1, 1)$, $A_2 = (2, -1, 1, 1)$, $A_3 = (2, -7, -1, -1)$.

The first summand is called the orthogonal projection of X; the second the orthogonal component of X.

907 Assume the vectors $A_1, A_2, \cdots, A_m$ are independent. Find a general expression for the orthogonal component of a vector X with respect to $A_1, A_2, \cdots, A_m$.

908 Show that of all vectors in a given subspace P, the one that makes the smallest angle with the vector X is the orthogonal projection of X onto the subspace P.

909 Let the subspace P be spanned by the vectors A_1, A_2 (or A_1, A_2, A_3). Find the smallest possible angle between the given vector x and any vector in the space P.

a) $X=(1, 3, -1, 3)$, $A_1=(1, -1, 1, 1)$, $A_2=(5, 1, -3, 3)$;
b) $X=(2, 2, -1, 1)$, $A_1=(1, -1, 1, 1)$, $A_2=(-1, 2, 3, 1)$, $A_3=(1, 0, 5, 3)$.

910 Let an m-dimensional coordinate subspace be given; find the smallest possible angle made by any vector in the given subspace with the vector $(1, 1, \cdots, 1)$.

911 Let a given subspace P be given and let Y run through the vectors of this subspace. Show that the vector X - Y has the smallest possible length

when Y is the vector X', the orthogonal projection of X on P (this smallest possible length is called the distance from the point X to the subspace P).

912 Find the distance from the point X to the linear manifold $A_0 + t_1A_1 + \cdots + t_mA_m$:

a) $X=(1, 2, -1, 1)$, $A_0=(0, -1, 1, 1)$, $A_1=(0, -3, -1, 5)$, $A_2=(4, -1, -3, 3)$;
b) $X=(0, 0, 0, 0)$, $A_0=(1, 1, 1, 1)$, $A_1=(1, 2, 3, 4)$.

913 In the space of polynomials with real coefficients and degree less than or equal to n, the scalar product of the polynomials f_1, f_2 is defined as $\int_0^1 f_1(x) f_2(x)\,dx$. Find the distance from the origin to the manifold determined by the polynomial $x^n + a_1x^{n-1} + \ldots + a_n$.

914 Give a procedure for finding the shortest distance between two linear manifolds $X_0 + P$, $Y_0 + Q$, that is between two points in these respective manifolds.

915 See problem 897. The vertices of an n-dimensional regular simplex with edge length 1 are separated into two collections of m + 1 and n - m vertices, respectively. Through each collection, a linear manifold is drawn with minimum possible dimension. Find the shortest distance between the points of these manifolds, and locate the points between which this distance is realized.

*916 Two planes are determined by the pairs of vectors $A_1, A_2; B_1, B_2$ in four-dimensional space. Find the smallest possible angle formed between pairs of vectors, one from the first plane and one from the second.

a) $A_1 = (1, 0, 0, 0)$, $A_2 = (0, 1, 0, 0)$, $B_1 = (1, 1, 1, 1)$, $B_2 = (2, -2, 5, 2)$;

b) $A_1 = (1, 0, 0, 0)$, $A_2 = (0, 1, 0, 0)$, $B_1 = (1, 1, 1, 1)$, $B_2 = (1, -1, 1, -1)$.

*917 A three-dimensional <u>hyperplane</u> is drawn through the center of a four-dimensional cube and orthogonal to a diagonal. Describe the solid of intersection.

*918 If $B_1, B_2, \cdots, B_m$ is a set of m linearly independent vectors, the parallelipippedon determined by these vectors is defined as the set of all points of the form

$$t_1B_1 + t_2B_2 + \cdots + t_mB_m, \quad 0 \le t_1 \le 1, \cdots, 0 \le t_m \le 1.$$

Define inductively the volume of the parallelipippedon as the product of the volume of the <u>base</u> $[B_1, B_2, \cdots, B_{m-1}]$ by the <u>altitude</u>, the latter being the distance from the vector B_m to the space spanned by the base. To begin the induction define the volume of a parallelipippedon $[B_1]$ consisting of a single vector as the length of the vector.

a) Give a formula for the square of the volume and deduce that the value of the volume is independent of the ordering of the vertices.

b) Show that $V[cB_1, B_2, \ldots, B_m] = |c| \cdot V[B_1, B_2, \ldots, B_m]$.

c) Show that $V[B_1' + B_1'', B_2, \ldots, B_m] \leqslant V[B_1', B_2, \ldots, B_m] + V[B_1'', B_2, \ldots, B_m]$, and find when the inequality degenerates to an equality.

919 Show that the volume of an n-dimensional parallelipippedon in n-dimensional space is the same as the absolute value of the determinant of the matrix whose rows are the coordinates of the generating vectors.

*920 Let $C_1, C_2, \cdots, C_m$ be the orthogonal projections of the vectors $B_1, B_2, \cdots, B_m$ onto some space. Prove that

$$V[C_1, C_2, \ldots, C_m] \leqslant V[B_1, B_2, \ldots, B_m].$$

*921 See problem 518. Show that

$$V[A_1, A_2, \ldots A_m, B_1, \ldots, B_k] \leqslant V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k]$$

922 See problem 519. Show that

$$V[A_1, A_2, \ldots, A_m] \leqslant |A_1| \cdot |A_2| \ldots |A_m|$$

923 Find the volume of the n-dimensional sphere by using Cavalieri's principle.

924 The scalar product of two polynomials of degree $\leq n$ is defined as $\int_0^1 f_1(x) f_2(x)\, dx$. Find the volume of the parallelipippedon generated by vectors defined as the sequence of coefficients of a set of polynomials.

3. PROPER VALUES AND PROPER VECTORS OF A MATRIX.

925 Find the proper values and proper vectors of the following matrices:

$$\text{a) } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \quad \text{b) } \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}; \quad \text{c) } \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}; \quad \text{d) } \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$

$$\text{e) } \begin{pmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}; \quad \text{f) } \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}; \quad \text{g) } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\text{h) } \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}; \quad \text{i) } \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}; \quad \text{j) } \begin{pmatrix} 2 & 5 & -6 \\ 4 & 6 & -9 \\ 3 & 6 & -8 \end{pmatrix}.$$

926 Find the connection between the proper values of the matrix A, and those of its inverse A^{-1}.

927 Find the relation between the proper values of the matrix A, and those of its square A^2.

928 Find the proper values of the matrix A^m in terms of the proper values of the matrix A.

929 Let $f(\lambda)$ be the characteristic polynomial of an n-th order matrix A. Let $f(x) = b_0(x-\xi_1)(x-\xi_2)\dots(x-\xi_m)$. Find the determinant of the matrix f(A).

930 Let f(x) be a given polynomial. Evaluate the determinant of the matrix f(A), in terms of the proper values of the matrix A.

931 Use the result of the preceding problem to find the proper values of the matrix f(A) in terms of the proper values of the matrix A.

932 Show that every proper vector of the matrix A is also a proper vector of the matrix f(A), where f is any polynomial.

*933 Find the proper values of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{pmatrix},$$

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, and n is odd.

*934 With ε define as in the preceding problem, evaluate the sum

$$1 + \varepsilon + \varepsilon^4 + \dots + \varepsilon^{(n-1)^2}.$$

935 Find the proper values of the following matrices:

a) $\begin{pmatrix} 0 & x & x & \dots & x \\ y & 0 & x & \dots & x \\ y & y & 0 & \dots & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y & y & y & \dots & 0 \end{pmatrix}$; b) $\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_2 & a_3 & \dots & a_1 \end{pmatrix}$;

c) $\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$,

*936 See problem 534. Show how the proper values of the Kronecker product of two matrices can be found in terms of the proper values of the factors.

937 Let A, B be arbitrary square matrices. Show that AB and BA have the same characteristic polynomial.

938 Let A, B be rectangular matrices with m rows and n columns [n rows and m columns] respectively, $n > m$. Show that the characteristic polynomials of AB, BA differ only in the factor $(-\lambda)^{n-m}$.

Reference: H. Flanders, Elementary Divisors of AB and BA, Proceedings of the American Mathematical Society, 2 (1951) 871-874.

4. QUADRATIC FORMS AND SYMMETRIC MATRICES

939 Transform each of the following into a sum of squares:

a) $x_1^2+2x_1x_2+2x_2^2+4x_2x_3+5x_3^2$;
b) $x_1^2-4x_1x_2+2x_1x_3+4x_2^2+x_3^2$;
c) $x_1x_2+x_2x_3+x_3x_1$;
d) $x_1^2-2x_1x_2+2x_1x_3-2x_1x_4+x_2^2+2x_2x_3-4x_2x_4+x_3^2-2x_4^2$;
e) $x_1^2+x_1x_2+x_3x_4$.

940 Transform the following into diagonal form:

$$\sum_{i=1}^{n} x_i^2 + \sum_{i<k} x_i x_k$$

941 Transform the following into diagonal form:

$$\sum_{i<k} x_i x_k.$$

942 Show that every principal minor of a positive definite quadratic form is positive.

943 Suppose that the quadratic form

$$\begin{aligned} f = a_{11}x_1^2 &+ a_{12}x_1x_2 + \dots + a_{1n}x_1x_n \\ + a_{21}x_2x_1 &+ a_{22}x_2^2 + \dots + a_{2n}x_2x_n \\ &\cdots\cdots\cdots\cdots \\ + a_{n1}x_nx_1 &+ a_{n2}x_nx_2 + \dots + a_{nn}x_n^2 \end{aligned}$$

can be transformed into diagonal form $a_1x_1'^2 + a_2x_2'^2 + \dots + a_nx_n'^2$ by the triangular transformation

$$\begin{aligned} x_1' &= x_1 + b_{12}x_2 + \dots + b_{1n}x_n \\ x_2' &= \phantom{x_1 + b_{12}} x_2 + \dots + b_{2n}x_n \\ &\cdots\cdots\cdots\cdots \\ x_n' &= \phantom{x_1 + b_{12}x_2 + \dots + b_{2n}} x_n. \end{aligned}$$

Find:

a) an expression for the coefficients $a_1, a_2, \cdots, a_n$ in terms of the coefficients a_{ik};

b) the discriminants of the form

$$f_k(x_{k+1}, \dots, x_n) = f - a_1'x_1'^2 - \dots - a_kx_k'^2$$

in terms of the coefficients a_{ik}.

Find conditions under which a triangular transformation of the given type exists.

944 (Sylvester's conditions) Show that the quadratic form

$$\begin{aligned} f = {} & a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{1n}x_1x_n \\ & + a_{21}x_2x_1 + a_{22}x_2^2 + \dots + a_{2n}x_2x_n \\ & \dots\dots\dots\dots\dots\dots\dots \\ & + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \dots + a_{nn}x_n^2 \end{aligned}$$

is positive definite if and only if the following inequalities are valid:

$$a_{11} > 0; \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0; \dots; \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} > 0.$$

*945 Show that if the square of a linear form is added to a positive definite quadratic form the discriminant is increased.

*946 Let the quadratic form $f(x_1, x_2, \dots, x_n) = a_{11}x_1^2 + \dots$ be positive definite; suppose

$$\varphi(x_2, \dots, x_n) = f(0, x_2, \dots, x_n),$$

and let D_f , D_φ be the discriminants of f and φ respectively. Show that the relation

$$D_f \leqslant a_{11} D_\varphi$$

holds.

947 Let $l_1, l_2, \ldots, l_p, l_{p+1}, l_{p+2}, \ldots, l_{p+q}$ be real linear forms in $x_1, x_2, \cdots, x_n$. Show that the number of squares in the canonical representation of the form

$$f(x_1, x_2, \ldots, x_n) = \\ = l_1^2 + l_2^2 + \ldots + l_p^2 - l_{p+1}^2 - l_{p+2}^2 - \ldots - l_{p+q}^2$$

cannot exceed p, and the number of negative squares cannot exceed q.

*948 Let s_0, s_1, be the sums of the powers of the roots of the equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, the coefficients of which are real. Show that the number of negative squares in the canonical representation of the quadratic form $\sum_{i,k=1}^{n} s_{i+k+2} x_i x_k$ is equal to the number of pairs of complex conjugate roots of the given equation.

Prove the following theorems:

949 If all roots of a polynomial equation with real coefficients are real and distinct, then the inequalities

$$\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix} > 0; \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} > 0; \ldots; \Delta = \begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \ldots & s_{2n-2} \end{vmatrix} > 0.$$

will hold, and conversely.

*950 If the forms

$$\begin{aligned} f = {} & a_{11}x_1^2 + a_{12}x_1x_2 + \ldots + a_{1n}x_1x_n \\ & + a_{21}x_2x_1 + a_{22}x_2^2 + \ldots + a_{2n}x_2x_n \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \ldots + a_{nn}x_n^2 , \end{aligned}$$

$$\begin{aligned} \varphi = {} & b_{11}x_1^2 + b_{12}x_1x_2 + \ldots + b_{1n}x_1x_n \\ & + b_{21}x_2x_1 + b_{22}x_2^2 + \ldots + b_{2n}x_2x_n \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + b_{n1}x_nx_1 + b_{n2}x_nx_2 + \ldots + b_{nn}x_n^2 \end{aligned}$$

are non-negative definite, then the form

$$\begin{aligned} (f, \varphi) = {} & a_{11}b_{11}x_1^2 + a_{12}b_{12}x_1x_2 + \ldots + a_{1n}b_{1n}x_1x_n \\ & + a_{21}b_{21}x_2x_1 + a_{22}b_{22}x_2^2 + \ldots + a_{2n}b_{2n}x_2x_n \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + a_{n1}b_{n1}x_nx_1 + a_{n2}b_{n2}x_nx_2 + \ldots + a_{nn}b_{nn}x_n^2 \end{aligned}$$

will be non-negative definite.

951 Use an orthogonal transformation to transform each of the following into canonical form:

a) $2x_1^2 + x_2^2 - 4x_1x_2 - 4x_2x_3$;
b) $x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3$;
c) $3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$;
d) $2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$;
e) $x_1^2 - 2x_2^2 - 2x_3^2 - 4x_1x_2 + 4x_1x_3 + 8x_2x_3$;
f) $5x_1^2 + 6x_2^2 + 4x_3^2 - 4x_1x_2 - 4x_1x_3$;
g) $3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 - 8x_1x_3 - 4x_2x_3$;

h) $7x_1^2+5x_2^2+3x_3^2-8x_1x_2+8x_2x_3$;

i) $2x_1^2+2x_2^2+2x_3^2+2x_4^2-4x_1x_2+2x_1x_4+2x_2x_3-4x_3x_4$;

j) $2x_1x_2+2x_3x_4$;

k) $x_1^2+x_2^2+x_3^2+x_4^2+2x_1x_2-2x_1x_4-2x_2x_3+2x_3x_4$;

l) $2x_1x_2+2x_1x_3-2x_1x_4-2x_2x_3+2x_2x_4+2x_3x_4$;

m) $x_1^2+x_2^2+x_3^2+x_4^2-2x_1x_2+6x_1x_3-4x_1x_4$
$-4x_2x_3+6x_2x_4-2x_3x_4$;

n) $8x_1x_3+2x_1x_4+2x_2x_3+8x_2x_4$.

952 Use an orthogonal transformation to transform each of the following into canonical form:

$$\text{a) } \sum_{i=1}^{n} x_i^2+\sum_{i<k} x_ix_k; \quad \text{b) } \sum_{i<k} x_ix_k.$$

953 Use an orthogonal transformation to transform the following into canonical form:

$$x_1x_2+x_2x_3+\dots+x_{n-1}x_n.$$

954 Suppose that all proper values of the real matrix A lie in the segment [a, b]. Show that the quadratic form having matrix A - λ**E** is negative definite for λ > b and positive definite for λ < a. The converse is also true.

955 Suppose that all the proper values of the real symmetric matrices a, b lie on the segments [a, c]; [b, d] respectively. Show that all the

proper values of the matrix $A + B$ lie in the segment $[a + b, c + d]$.

956 Let A be a real matrix, A' its transpose. By the norm of A (denoted $\|A\|$) we mean the positive square root of the greatest proper value of the matrix $A'A$. Establish the following

a) $\|A\| = \|A'\|$;

b) Length $(AX) \leq \|A\| \cdot$ Length (X), and there is a vector X_0 for which equality occurs;

c) $\|A + B\| \leq \|A\| + \|B\|$;

d) $\|AB\| \leq \|A\| \cdot \|B\|$;

e) Every proper value of the matrix A is less than or equal to $\|A\|$ in absolute value.

957 Show how to represent an arbitrary real nonsingular matrix as the product of an orthogonal matrix by a triangular matrix of the form

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & \cdots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix};$$

in the latter the diagonal elements b_{ii} are positive. This representation is unique.

958 Show that a real nonsingular matrix can be written as the product of an orthogonal matrix by a symmetric matrix, the latter corresponding to a positive definite quadratic form.

959 Let the second degree surface S in n-dimensional space be defined by the equation

$$\begin{aligned}
&a_{11}x_1^2 + a_{11}x_1x_2 + \dots + a_{1n}x_1x_n \\
&+ a_{21}x_2x_1 + a_{22}x_2^2 + \dots + a_{2n}x_2x_n \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&+ a_{n1}x_nx_1 + a_{n2}x_nx_2 + \dots + a_{nn}x_n^2 \\
&+ 2b_1x_1 + 2b_2x_2 + \dots + 2b_nx_n + c = 0,
\end{aligned}$$

or in condensed notation $AX \cdot X + 2B \cdot X + c = 0$. Show that a necessary and sufficient condition that the surface S have a center is that rank A = rank (A, B).

960 See the preceding exercise. Let S be a second degree central surface. Show that the equation of S can be reduced to the form

$$\alpha_1x_1^2 + \dots + \alpha_rx_r^2 + \gamma = 0$$

by a translation followed by a rotation (orthogonal transformation).

961 See problem 960. If S is a non-central surface of degree two, show that its equation can be reduced to the form

$$\alpha_1x_1^2 + \dots + \alpha_rx_r^2 = 2x_{r+1}$$

by a translation followed by a rotation.

5. LINEAR SPACES. THE JORDAN CANONICAL FORM.

962 Let A be a linear transformation defined on a space of dimension n. Show that the dimension of the range of A (the set of all vectors y of the form $y = Ax$) is equal to the rank of A.

963 Let Q be a q-dimensional subspace of the n-dimensional space R; let A be a linear transformation of R of rank r; let Q' be the image of Q. If q' is the dimension of Q', show that the following relation holds:

$$q + r - n \leq q' \leq \min(q, r).$$

964 Use the result of problem 963 to show that the rank ρ of the product of two matrices of ranks r_1, r_2 satisfies the inequalities

$$r_1 + r_2 - n \leq \rho \leq \min(r_1, r_2).$$

*965 Let Q, Q' be arbitrary complementary subspaces of the space P. Then every vector $X \in P$ can be represented uniquely as the sum of vectors $Y \in Q$, $Y' \in Q'$. For each vector X a linear transformation can be defined carrying X into its component Y in Q; this transformation is called a projection onto Q parallel to Q'. Prove that this transformation is a linear transformation and show that the matrix of this transformation satisfies the relation $A^2 = A$.

Does this last assertion depend on the particular basis chosen? Conversely, if A is a linear transformation satisfying the relation $A^2 = A$, then it corresponds to a projection operator.

*966 A projection is called orthogonal if the complementary spaces Q', Q are orthogonal. Show that the matrix of a projection is symmetric whenever the base is orthonormal. Conversely every symmetric idempotent matrix ($A^2 = A$) is the matrix of an orthogonal projection.

*967 Show that every non-null proper value of a skew-symmetric matrix is pure imaginary; the real and imaginary parts of the corresponding proper vector have the same length and are mutually orthogonal.

*968 Show that the skew-symmetric matrix A can be transformed by a suitable orthogonal matrix P as follows:

$$P^{-1}AP = \begin{pmatrix} 0 & a_1 & & & & & & & & \\ -a_1 & 0 & & & & & & & & \\ & & 0 & a_2 & & & & & & \\ & & -a_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & a_k & & & \\ & & & & & -a_k & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{pmatrix}.$$

Here $a_1, a_2, \cdots, a_k$ are real and all numbers not explicitly written are zero.

969 Establish the following theorem: Let A be a skew-symmetric matrix. Then the matrix $(E - A)(E + A)^{-1}$ is orthogonal and does not have -1 as a proper value. Conversely every orthogonal matrix which does not have -1 as a proper value can be written in this form.

*970 Show that every proper value of an orthogonal matrix has absolute value 1.

*971 Show that every proper vector of an orthogonal matrix corresponding to a nonreal proper value can be written in the form X + iY, where X, Y are real, have the same length, and are orthogonal.

*972 Show that every orthogonal matrix can be written in the form $Q^{-1}TQ$, where Q is orthogonal, and T has has the form

$$\begin{pmatrix}
\cos\varphi_1, -\sin\varphi_1 & & & & & & & & & & & & \\
\sin\varphi_1 \quad \cos\varphi_1 & & & & & & & & & & & & \\
& \cos\varphi_2, -\sin\varphi_2 & & & & & & & & & & & \\
& \sin\varphi_2 \quad \cos\varphi_2 & & & & & & & & & & & \\
& & \cdot & & & & & & & & & & \\
& & & \cdot & & & & & & & & & \\
& & & & \cdot & & & & & & & & \\
& & & & & 1 & & & & & & & \\
& & & & & & 1 & & & & & & \\
& & & & & & & \cdot & & & & & \\
& & & & & & & & \cdot & & & & \\
& & & & & & & & & \cdot & & & \\
& & & & & & & & & & 1 & & \\
& & & & & & & & & & & -1 & \\
& & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & -1 \\
& & & & & & & & & & & & & & -1
\end{pmatrix}$$

973 Transform each of the following matrices into Jordan canonical form:

a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$; b) $\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$;

c) $\begin{pmatrix} 13 & 16 & 16 \\ -5 & -7 & -6 \\ -6 & -8 & -7 \end{pmatrix}$; d) $\begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & -0 & -5 \end{pmatrix}$;

e) $\begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix}$; f) $\begin{pmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{pmatrix}$;

g) $\begin{pmatrix} -2 & 8 & 6 \\ -4 & 10 & 6 \\ 4 & -8 & -4 \end{pmatrix}$; h) $\begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$;

i) $\begin{pmatrix} -1 & 1 & 1 \\ -5 & 21 & 17 \\ 6 & -26 & -21 \end{pmatrix}$; j) $\begin{pmatrix} 8 & 30 & -14 \\ -6 & -19 & 9 \\ -6 & -23 & 11 \end{pmatrix}$;

k) $\begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$; l) $\begin{pmatrix} 3 & 7 & -3 \\ -2 & -5 & 2 \\ -4 & -10 & 3 \end{pmatrix}$,

m) $\begin{pmatrix} 9 & 22 & -6 \\ -1 & -4 & 1 \\ 8 & 16 & -5 \end{pmatrix}$; n) $\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$; o) $\begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix}$.

974 Transform each of the following matrices into Jordan canonical form:

$$\text{a)}\begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 7 & 1 & 2 & 1 \\ -17 & -6 & -1 & 0 \end{pmatrix};\quad \text{b)}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix};\quad \text{c)}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

*975 Let the matrix A have finite order: $A^m = E$ for some positive integer m. Show that the canonical form of A is diagonal.

*976 See problem 531. Find the proper values of the matrix $A_{(m)}$ in terms of the proper values of the matrix A.

977 Show how to transform an arbitrary matrix A into its transpose.

*978 Show that every matrix can be transformed into the product of two symmetric matrices, one of which is nonsingular.

979 Starting with a given n-th order matrix A, construct a sequence of matrices according to the following rule:

$$\begin{array}{llll}
A_1 = A; & \mathrm{Sp}A_1 = p_1; & A_1 - p_1E = B_1, \\
B_1A = A_2; & \frac{1}{2}\,\mathrm{Sp}A_2 = p_2; & A_2 - p_2E = B_2, \\
B_2A = A_3; & \frac{1}{3}\,\mathrm{Sp}A_3 = p_3; & A_3 - p_3E = B_3, \\
\cdots & \cdots & \cdots \\
B_{n-1}A = A_n; & \frac{1}{n}\,\mathrm{Sp}A_n = p_n; & A_n - p_nE = B_n.
\end{array}$$

Show that the numbers $p_1, p_2, \cdots, p_n$ are the coefficients of the characteristic polynomial

$$(-1)^n[\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \ldots - p_n]$$

of A. Show that the matrix B_n is the null matrix. Finally show that if A is nonsingular, then the relation $\frac{1}{p_n}B_{n-1} = A^{-1}$ holds.

980 Let C be a matrix for which the equation XY - YX = C has a solution in square matrices X, Y. Then the trace (the sum of the diagonal elements) of C is zero. Prove the converse of this theorem.

981 Let C be a matrix of complex elements, C^* its hermitian conjugate (conjugate of the transpose). If all singular values ρ_i of C satisfy $|\rho_i| < 1$, show that a positive definite hermitian matrix H can be found $[H = H^*;\ x \neq 0 \to x\,Hx^* > 0]$ such that $H - CHC^* = I$, identity matrix.

A <u>singular</u> <u>value</u> of C is a proper value of CC^*.

982 If C is the matrix of problem 981, and if B is an arbitrary positive definite hermitian matrix $[B = B^*;\ x \neq 0 \longrightarrow x\,Bx^* > 0]$, a positive definite hermitian matrix H can be found such that

$$H - CHC^* = B.$$

983 Problem 982 uses the (strong) hypothesis that all proper values of CC^* have modulus less than 1. In point of fact, the conclusion remains valid under the (weaker) hypothesis that all proper values of C have modulus less than 1. Assume this proved, and from it, establish the Lyapunov theorem: If R is an arbitrary positive-definite hermitian matrix and A is a stable matrix (i.e. a matrix, all proper values of which have negative real part), there is a positive-definite hermitian matrix G such that

$$AG + GA^* = -R.$$

CHAPTER I - HINTS

COMPLEX NUMBERS

11 See problem 10.

13 First establish the theorem for each of the four binary operations; then use mathematical induction.

18 Assume that the left member is the sum of two squares.

27 Set $x = a + bi$, $y = c + di$.

28 Set $z = \cos \varphi + i \sin \varphi$.

31 Set $z = t^2$, $z' = t'^2$. Use problem 27.

37 Use polar form.

38 $1 + \omega = -\omega^2$.

40 Use the half angle formulas.

41 Use Demoivre's theorem and note the following

$$z = \cos\theta \pm i \sin\theta; \quad \frac{1}{z} = \cos\theta \mp i \sin\theta.$$

51 Let $\alpha = \cos x + i \sin x$. Then

$$\cos^{2m} x = \left(\frac{\alpha + \alpha^{-1}}{2}\right)^{2m}$$

52 Show that the coefficient of $(2 \cos x)^{m-2p}$ is $(-1)^p(C_p^{m-p} + C_{p-1}^{m-p-1})$. Use mathematical induction.

53 This problem is similar to the preceding.

54 Use Newton's formula for the binomial expansion of $(1 + i)^n$.

55 Similar to the preceding; choose the proper binomial.

56 In this case the binomial to choose is $\left(1+i\frac{\sqrt{3}}{3}\right)^n$.

68 Show that the problem comes down to the calculation of the limit of the sum $1 + \alpha + \alpha^2 + \cdots$, where $\alpha = \frac{-1 + i}{2}$.

69 Use the formula $\sin^2\alpha = \frac{1}{2} - \frac{\cos 2\alpha}{2}$.

71 Use the formulas

$$\cos^3\alpha = \frac{\cos 3\alpha}{4} + \frac{3\cos\alpha}{4}; \quad \sin^3\alpha = \frac{3\sin\alpha}{4} - \frac{\sin 3\alpha}{4}.$$

72 To calculate sums of the form $1 + 2a + 3a^2 + 4a^3 + \cdots + na^{n-1}$, $1 + 2^2a + 3^2a^2 + \cdots + n^2a^{n-1}$ it is convenient to multiply first by the factor $1 - a$.

76 $x_1 = \alpha + \beta; \quad x_2 = \alpha\omega + \beta\omega^2; \quad x_3 = \alpha\omega^2 + \beta\omega;$

$$\alpha^3 + \beta^3 = -q; \quad 3\alpha\beta = -p.$$

77 Multiply by -27 and think of the left member as the discriminant of some cubic equation.

78 Set $x = \alpha + \beta$.

87 Use the fact that $\varepsilon^n = -1$.

88 If $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, note that the sum in question has the form $1 + \varepsilon + \varepsilon^2 + \ldots + \varepsilon^{n-1}$.

89 Consider two cases: 1) k divides n; 2) k does not divide n.

91, 92 Multiply by $1 - \varepsilon$.

94 a) Start with all 15 roots of unity; subtract the sum of the roots of degrees 1, 3, 5.

97 If a 14-sided figure is inscribed on a circle of radius 1, the length of its side is $2 \sin \pi/14$. Use the additional fact that the equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ has the number $\cos 4\pi/7 + i \sin 4\pi/7$ as one root.

98 1) If $x_1, x_2, \cdots, x_n$ are roots of the equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_n = 0, \quad \text{then}$$

$$a_0x^n + a_1x^{n-1} + \ldots + a_n = a_0(x - x_1) \ldots (x - x_n);$$

2) If ε is an n-th root of 1, then the complex conjugate is also an n-th root of 1.

99 Use the result of problem in 98 in the special case $x = 1$.

100 Use the identity showing the factorization of $x^n - 1$ into first degree factors.

101 Factor $x^n - 1$ into factors of the first degree; then set

1) $x = \cos\theta + i \sin\theta$;

2) $x = \cos\theta - i \sin\theta$.

103 Use the fact that two conjugate complex numbers have the same modulus.

105 a) Transform the equation into the form

$$\left(\frac{x+1}{x-1}\right)^m = 1.$$

107 Let

$$S = \cos\varphi + C_1^n \cos(\varphi+\alpha)\,x + \ldots + \cos(\varphi+n\alpha)\,x^n,$$
$$T = \sin\varphi + C_1^n \sin(\varphi+\alpha)\,x + \ldots + \sin(\varphi+n\alpha)\,x^n.$$

Calculate S + Ti, S - Ti and determine S from the results of the calculation.

113 First show that $\varphi(p^\alpha) = p^\alpha\left(1 - \frac{1}{p}\right)$, if p is a prime number. This can be done by counting up all the positive integers less than are equal to p^α and not divisible by p. Then use 112.

116 Show that the only primitive roots of $x^{p^m} - 1$ are those that are not simultaneously roots of $x^{p^{m-1}} - 1$.

117 Show that if n is odd, every primitive 2n-th root of unity is the negative of a primitive n-th root of -1; there are no others.

119 Use 118.

120 Use 115, 116, 111 and establish the following

1) $\mu(p) = -1$, when p is prime;
2) $\mu(p^\alpha) = 0$, when p is prime and $\alpha > 1$;
3) $\mu(ab) = \mu(a)\mu(b)$, if a, b are relatively prime.

122 Show that if $\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ corresponds to the value of n_1, then $x - \epsilon_k$

appears in the right member of the equation to the power $\sum \mu(d_1)$, where d_1 runs through all divisors of n/n_1.

123 Consider two cases: 1) n is a power of a prime number; 2) n is the product of powers of different primes. For case 1 use problem 116; for case 2 problems 119, 122.

124 Consider four cases:

1) n odd greater than 1;
2) n a power of 2;
3) $n = 2n_1$, where n_1 is odd and greater than 1;
4) $n = 2^k n_1$, where k is greater than 1, n_1 is odd and greater than 1.

125 Use the identity

$$x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n = \frac{(x_1 + x_2 + \dots + x_n)^2 - (x_1^2 + x_2^2 + \dots + x_n^2)}{2}.$$

Separate into three cases:

1) n odd;
2) n twice an odd number;
3) $n = 2^k n_1$, where k greater than 1, n_1 odd.

126 Multiply the sum S by its conjugate and note that ε^{x^2} does not change if x is replaced by $x + n$.

CHAPTER II - HINTS

COMPUTATION OF DETERMINANTS

132 Every pair of numbers is reversed.

133 The number of inversions in the second permutation is equal to the number of sequences in the first in natural order.

145 Show that every term contains the factor 0.

149, 150 Interchange rows and columns.

153 Note how the value of the determinant changes if an arbitrary permutation is performed on the columns

154 a) Use the factor theorem and note that, if $x = a_1$, the determinant has two equal rows.

155 Add 100 times the first column plus 10 times the second column to the third column.

156 First subtract the first column from each of the others.

163 Subtract the first column from the second.

179 Add the sum of the last $n - 1$ rows to the first.

180, 181, 182 Subtract the first row from each of the others.

183 Subtract the second row from each of the others.

184 Add the first row to the second.

185 Add the sum of all the remaining columns to the first.

186, 187 Modify the first column as follows: subtract the second, add the third, etc.

188 Expand according to the minors of the elements of the first column; or add to the last column x^n times the first column + x^{n-1} times the second column + $\cdots$

189 Add x^{n-1} times the first column + x^{n-2} times the second column + $\cdots$ to the last column.

190 First set up the matrix whose determinant is needed; then proceed as follows: add to the last column, -1 times the first, $-x$ times the second, $-x^2$ times the third, etc.

191 Subtract a_1 times the last column from the first; a_2 times the last column from the second; $\cdots$ a_n times the last column from the n-th.

192 Add the sum of the remaining columns to the first.

194 Add the sum of the remaining columns to the last.

195 Factor a_1 from the first column, a_2 from the second, a_3 from the third,...,a_n from the last. Then add all the columns to the last column.

196 Factor h from the first column; add the first column to the second.

197 Multiply the first row and the first column by x; add all the remaining rows to the first; make further transformations of a similar type.

198 To the second row add $-a_1$ times the first; to the third row add $-a_2$ times the first; $\cdots$. Perform a similar transformation on the columns.

199, 200 Add the remaining columns to the first.

201 First add $-a$ times the next last column to the last; then add $-a$ times the preceding column to the next last column; $\cdots$.

202 Similar to 201 using the rows; the factor a is replaced by 1. Finally add the first column to each of the others.

203 Multiply the first row by b_0, the second by b_1 and so on; finally add the sum of the last $n - 1$ rows to the first.

204 Factor a from the first row; subtract the first row from the second.

205 Expand by minors of the elements of the first row.

206 Write as the sum of 2 determinants.

208 Imagine 0 as the third summand of every term not on the principal diagonal; write the determinant as the sum of 2^n determinants. Use problems 206, 207.

211 Multiply the first column by x^{n-1}, the second by x^{n-2} and so on.

212 Expand according to minors of the last column, and note the inductive relation

$$\Delta_n = x_n\Delta_{n-1} + a_n x_1 x_2 \dots x_{n-1}$$

where Δ_n is the value of the n-th order determinant. Finally use mathematical induction.

213 Expand according to minors of the elements of the last row and show that $\Delta_{n+1} = x_n\Delta_n + a_n y_1 y_2 \dots y_n$. See the hint for number 212.

214 Factor a_1 from the second column; a_2 from the third; $\cdots$; a_n from the $(n + 1)$-st. Reverse the sign of the first column and add all the remaining columns to the first.

215 Expand according to minors of the elements of the first row.

216 Expand according to minors of the elements of the first row and show that

$$\Delta_n = a_1 a_2 \dots a_{n-1} - \Delta_{n-1}.$$

See the hint to problem 212.

219 Use the result of problem 217.

221 Expand according to the elements of the first row and note that

$$\Delta_n = x\Delta_{n-1} - \Delta_{n-2}.$$

Use the hint to problem 212.

222 Subtract y_n/y_{n-1} times the next last row from the last. Establish the recursion formula

$$\Delta_n = \frac{y_n}{y_{n-1}}(x_n y_{n-1} - x_{n-1} y_n)\Delta_{n-1}.$$

See the hint to number 212.

223 Express as the sum of two determinants and establish the recursion formula

$$\Delta_n = a_n \Delta_{n-1} + a_1 a_2 \ldots a_{n-1}.$$

225 Express as the sum of two determinants and establish the recursion formula

$$\Delta_n = (a_n - x)\Delta_{n-1} + x(a_1 - x) \ldots (a_{n-1} - x).$$

226 Rewrite x_n in the form $x_n = (x_n - a_n) + a_n$; write the determinant as the sum of two determinants and establish the recursion formula

$$\Delta_n = (x_n - a_n)\Delta_{n-1} + a_n(x_1 - a_1)(x_2 - a_2) \ldots (x_{n-1} - a_{n-1}).$$

227 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = (x_n - a_n b_n)\Delta_{n-1} + a_n b_n (x_1 - a_1 b_1) \ldots (x_{n-1} - a_{n-1} b_{n-1}).$$

228 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = -m\Delta_{n-1} + (-1)^{n-1} m^{n-1} x_n.$$

230 Expand according to minors of the elements of the first row and show that

$$\Delta_{2n} = (a^2 - b^2)\Delta_{2n-2}.$$

231 From every row subtract the preceding; and add all the remaining rows to the second one. Finally expand the determinant according to minors of the elements of the last row and establish the recursion formula

$$\Delta_n = [a + (n-1)b]\,\Delta_{n-1} + a(a+b)\ \dots\ [a + (n-2)b].$$

See the hint for problem 212.

232 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = x(x - 2a_n)\,\Delta_{n-1} + a_n^2 x^{n-1} \prod_{i=1}^{n-1} (x - 2a_i).$$

233 Make the algebraic transformation $(x - a_n)^2 = x(x - 2a_n) + a_n^2$, write the determinant as the sum of two determinants and establish the recursion formula

$$\Delta_n = x(x - 2a_n)\,\Delta_{n-1} + a_n^2 x^{n-1} (x - 2a_1)\ \dots\ (x - 2a_{n-1}).$$

234 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = \Delta_{n-1} + (-1)^n b_1 b_2 \dots b_n.$$

235 Write the last element of the last row in the form $a_n - a_n$. Establish the recursion formula

$$\Delta_n = (-1)^{n-1} b_1 b_2 \dots b_{n-1} a_n - a_n \Delta_{n-1}.$$

236 Subtract the last row from each of the others.

237 Write the first element of the first row in the form $1 = x + (1 - x)$. Write the determinant as the sum of two determinants and use the result of 236.

238, 239 Multiply the second row by x^{n-1}; the third by x^{n-2}; $\cdots$; the n-th by x. Factor x^n from the first column, x^{n-1} from the second column; $\cdots$; x from the n-th column.

240 Starting at the last column, add -1 times the preceding column to each column. Then proceed similarly with the rows. Show that $\Delta_n = \Delta_{n-1}$. The subtractions are easily performed by noting the relation $C_k^n = C_k^{n-1} + C_{k-1}^{n-1}$.

241 Starting with the next to last, add -1 times each column to the succeeding.

242 From the n-th, $\cdots$ row subtract the preceding row; show that $\Delta_n = \Delta_{n-1}$.

243 Factor m from the first row; $m + 1$ from the second row; $\cdots$; $m + n$ from the last row. Factor $\frac{1}{k}$ from the first column; $\frac{1}{k+1}$ from the second column; $\cdots$ Continue this sequence of operations until all the elements of the first column become equal to 1.

244 First from each column subtract the preceding. In the resulting matrix change each column except the first two by subtracting the preceding column. Continue in this way, but leaving the first three,

the first four, etc. unaltered. After this operation has been performed m times, a matrix is obtained in which all the elements of the last column are 1. It is easy to compute its determinant.

245 From each row, subtract the preceding and establish the recursion formula

$$\Delta_{n+1} = (x - 1)\Delta_n.$$

See the hint for problem 212.

246 From each row subtract the preceding and establish the recursion formula

$$\Delta_{n+1} = (n - 1)!(x - 1)\Delta_n.$$

247 From each row subtract the preceding. From each column subtract the preceding. Establish the recursion formula $\Delta_n = \alpha\Delta_{n-1}$.

248 Write the last element in the last row in the form z + (x - z). Write the determinant as the sum of two determinants. At this point make the observation that the determinant is symmetric in y, z.

249 See the hint for the preceding problem.

252 Subtract β/a times the first row from each of the remaining rows. Factor the quantity

$$\frac{ab - \lambda\beta}{a(\alpha - \beta)}$$

from the first column and subtract all the remaining columns from the first.

253, 254 Add all the remaining columns to the first and from each row subtract the preceding. See problem 199.

256 Think of the determinant as a polynomial in a of the fourth degree. Show that each of the following linear factors divides the determinant:
$a + b + c + d$; $a + b - c - d$; $a - b + c - d$; $a - b - c + d$.

258 Add each of the last $n - 1$ columns to the first, and divide out the monomial $x + a_1 + \cdots + a_n$. By substituting $x = a_1, a_2, \cdots, a_n$ note that the determinant is divisible by each of
$x - a_1, x - a_2, \cdots, x - a_n$.

259 This is a Vandermonde determinant.

264 Expand according to minors of the elements of the first column.

265 Subtract the first column from the second; then subtract the second column from the third and so on.

269 Factor $\frac{1}{2!}$ from the third row; $\frac{1}{3!}$ from the fourth; $\cdots$

270 Use the result of problem 269.

271 Factor 2 from the second column; 3 from the third; $\cdots$ To evaluate the expression

$$\prod_{n \geqslant i > k \geqslant 1} (i^2 - k^2)$$

write it in the form

$$\prod (i^2 - k^2) = \prod (i - k) \cdot \prod (i + k).$$

272 Factor $\frac{x_1}{x_1-1}$ from the first column; $\frac{x_2}{x_2-1}$ from the second column; $\cdots$

273 Factor a_1^n from the first row; a_2^n from the second row; $\cdots$

275 To the first row add C_1^{2n} times the second row plus C_2^{2n} times the third row, etc.

276 Use the result of problem 51.

277 Use problem 53.

278 Border the matrix with the row $1, x_1, x_2, \cdots, x_n$ and the column $1, 0, 0, \cdots, 0$.

279 Border the matrix to obtain

$$D = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & z \\ x_1^2 & x_2^2 & \dots & x_n^2 & z^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^n & x_2^n & \dots & x_n^n & z^n \end{bmatrix}.$$

Find the determinant of D by expanding according to minors of the last column and obtain the result

$$\det D = \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k) \cdot \prod_{i=1}^{n} (z - x_i).$$

Find the coefficient of z in this product.

280 Use the hint for the preceding problem.

282 Border the matrix with first row 1, 0,...,0 and first column 1, 1, 1,···,1. Subtract the first column from each of the others.

285 Expand according to minors of the elements of the last row.

286 First from each column starting with the last subtract x times the preceding. Then reduce the order and factor out obvious monomial factors; transform the first row (that depends on x) by using the relation

$$(m+1)^s - m^s = sm^{s-1} + \frac{s(s-1)}{2} m^{s-2} + \dots + 1.$$

287 From each column beginning with the last subtract x times the preceding.

288 m) To the first column add the sixth and the eleventh; to the second column the seventh and the twelfth;···; to the fifth column the tenth and the fifteenth. Add the eleventh column to the sixth; the twelfth to the seventh;···; the fifteenth to the tenth.

Subtract the tenth row from the fifteenth; the ninth from the fourteenth;···; the first from the sixth.

293 Consider the product of the two matrices

$$\begin{bmatrix} 1 & C_1^n a_0 & C_2^n a_0^2 & \dots & a_0^n \\ 1 & C_1^n a_1 & C_2^n a_1^2 & \dots & a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & C_1^n a_n & C_2^n a_n^2 & \dots & a_n^n \end{bmatrix} \cdot \begin{bmatrix} b_0^n & b_1^n & \dots & b_n^n \\ b_0^{n-1} & b_1^{n-1} & \dots & b_n^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

294 Consider the product of the two matrices

$$\begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & 0 & \dots & 0 \\ \sin\alpha_2 & \cos\alpha_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sin\alpha_n & \cos\alpha_n & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \dots & \cos\alpha_n \\ \sin\alpha_1 & \sin\alpha_2 & \dots & \sin\alpha_n \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

295 Consider the product of the two matrices

$$\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^n & x_2^n & \dots & x_n^n & x^n \end{bmatrix} \cdot \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} & 0 \\ 1 & x_2 & \dots & x_2^{n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & \dots & x_n^{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

296 Square the matrix.

297 Subtract the first column from the third; the second column from the fourth; next multiply by the matrix

$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos 2\varphi & -\sin 2\varphi \\ 0 & 0 & \sin 2\varphi & \cos 2\varphi \end{bmatrix}.$$

298 Subtract n times the first column from the second; n times the second column from the fourth. Interchange the second and fourth columns. Finally multiply by the matrix

$$\begin{bmatrix} \cos n\varphi & -\sin n\varphi & 0 & 0 \\ \sin n\varphi & \cos n\varphi & 0 & 0 \\ 0 & 0 & \cos(n+1)\varphi & -\sin(n+1)\varphi \\ 0 & 0 & \sin(n+1)\varphi & \cos(n+1)\varphi \end{bmatrix}.$$

299 Square the matrix. Transform it as in evaluating the Vandermonde determinants; and write each difference as the sine of some angle. In this way determine the sign.

300 Consider the following matrix product

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \dots & a_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \varepsilon_1 & \dots & \varepsilon_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon_1^{n-1} & \dots & \varepsilon_{n-1}^{n-1} \end{bmatrix},$$

where $\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

308 Use the abbreviation $\varepsilon_1 = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$.

Then

$$\Delta = \prod_{k=0}^{n-1}\left(\sum_{j=1}^{n} \frac{\varepsilon_1^{j} + \varepsilon_1^{-j}}{2}\,\varepsilon_1^{2k(j-1)}\right).$$

311 Use problem 92.

314

$$\prod_{k=0}^{2n-1}(a_0+a_1\varepsilon_k+a_2\varepsilon_k^2+\ldots+a_{2n-1}\varepsilon_k^{2n-1})=$$

$$=\prod_{r=0}^{n-1}[(a_0+a_n)+(a_1+a_{n+1})\alpha_r+\ldots+(a_{n-1}+a_{2n-1})\alpha_r^{n-1}]$$

$$\times\prod_{s=0}^{n-1}[(a_0-a_n)+(a_1-a_{n+1})\beta_s+\ldots+(a_{n-1}-a_{2n-1})\beta_s^{n-1}],$$

where $\varepsilon_k=\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}$; $\alpha_r=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}$;

$$\beta_s=\cos\frac{(2s+1)\pi}{n}+i\sin\frac{(2s+1)\pi}{n}.$$

323 Subtract the first row from each of the others and the first column from each of the others.

325 Use problem 217.

327 Write the determinant as the sum of determinants or set x = 0 in the determinant and its derivatives.

328 1) From each row with index 2n - 1, 2n - 2,...,n + 1, subtract the preceding; from the n-th row subtract the sum of the others.

2) To the (n + i)-th row add the i-th, i = 1,2,···,n - 1.

329 To each row add the sum of all the remaining; from each column subtract the remaining. Obtain the recursion relation $\Delta_{n+1}(x)=(x-n)\Delta_n(x-1)$.

CHAPTER IV - HINTS

MATRICES

466 Use the result of problem 465 e).

473 Consider the sum of the elements in the principal diagonal.

491 Use the result of problems 489, 490.

492 Use the result of problem 490.

494, 495 Use the results of problems 492, 493.

496 Construct a proof by using induction on the number of columns of the matrix B. First show that if addition of one column does not change the rank of the matrix B, then it cannot change the rank the matrix (A, B). One can also establish the theorem from the Laplace expansion without using any induction.

497 Use the results of problems 496, 492.

498 Find a non-singular square submatrix D of the matrix (E - A, E + A) and consider the products (E - A)P, (E + A)P.

500 Use the result of problem 489.

501 Show that the representation in problem 500 is unique and reduce the problem to the enumeration of the number of triagonal matrices R with given determinant k. Calling this number $F_n(k)$, show that if k = ab, where a, b are relatively prime,

then $F_n(k) = F_n(a) \cdot F_n(b)$. Finally, use induction to obtain a formula for $F_n(p^m)$, where p is a prime number.

505 Use the results of problems 495, 498. Choose a matrix P with smallest possible determinant, and so that $P^{-1}AP$ is diagonal. Then use the result of problem 500.

517 Use Buniakovsky's inequality in Laplace's expansion.

518 If the sum of the products of the elements of an arbitrary column of B by the corresponding elements of an arbitrary column of C is 0, show that the equality det A'A = (det B'B)(det C'C) holds. Then border the matrix (D, C) so as to obtain a square matrix and use the result of problem 517.

523 Border the matrix with a column on the left all elements of which are M/2, and a row of n zeros. Then add the first column to each of the others.

527 Use the results of problems 522, 526.

528 Use the relation between the given matrix and its inverse.

529 Let A_{ik} be the algebraic cofactor of the element a_{ik}. To establish the result for the elements for the minor formed on the first m rows and the first m columns, consider the matrix product:

$$\begin{pmatrix} A_{11} & \dots & A_{m+1,1} & \dots & A_{n1} \\ A_{12} & \dots & A_{m+1,2} & \dots & A_{n2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{1m} & \dots & A_{m+1,m} & \dots & A_{nm} \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

The general result is obtained similarly.

535 Write $A \times B$ in the form $(A \times E_m) \cdot (E_n \times B)$.

537 Use an inductive proof starting with the case: A_{11} is a nonsingular matrix. The general case can be established by adding λE if necessary.

CHAPTER V - HINTS

POLYNOMIALS AND RATIONAL FUNCTIONS IN A SINGLE INDETERMINATE

547 a) First expand $f(x)$ in powers of $x - 3$; then replace x by $x + 3$.

553 In the derivative first set $x = 1$; then factor out as high a power of x as possible and differentiate again.

555 Introduce the sequence of auxiliary polynomials

$$f_1(x) = nf(x) - xf'(x); \quad f_2(x) = nf_1(x) - xf_1'(x), \cdots.$$

561 Establish the result by mathematical induction.

562 A non-zero $(k - 1)$-fold root of the polynomial $f(x)$ is a $(k - 2)$-fold root of the polynomial $x\,f'(x)$; a $(k - 3)$-fold root of the polynomial $x[x\,f'(x)]'$; $\cdots$

Conversely any nonzero root common to the polynomials $f(x)$, $x\,f'(x)$, $x[x\,f'(x)]'$ (a sequence of $k - 1$ polynomials) is a root of $f(x)$ of multiplicity at least $k - 1$.

563 Divide the polynomial by its derivative and differentiate the resulting equality.

567 Consider one of the functions $\frac{f_1(x)}{f_2(x)}$, $\frac{f_2(x)}{f_1(x)}$.

568 Let x_0 be a root of $[f'(x)]^2 - f(x)\,f''(x)$. Then examine the roots of $\varphi(x) = f(x)\,f'(x_0) - f'(x)\,f(x_0)$.

569 Expand $f(x)$ in powers of $x - x_0$; use the result of the preceding problem.

576 The proof is similar to the proof of D'Alembert's lemma.

580, 581 Write the function as follows (the form used in the proof of D'Alembert's lemma):

$$f(z) = f(a) + \frac{f^k(a)}{k!}(z-a)^k[1+\varphi(z)]; \qquad \varphi(a) = 0.$$

583 In a) and b) find the roots of the polynomial and factor the coefficient of the highest power of x. In c) the roots are conveniently found by making the substitution $x = tg^2\theta$.

589 Find the common roots.

608 As a first step show that f(x) has no real roots of odd multiplicity.

623 Use the result of problem 622.

626 Note that the equation is unchanged if x is replaced by -x or by 1/x.

627 Note that the equation is unchanged if x is replaced by 1/x or by 1 - x.

637 Factor out $(1 - x)^n$. Differentiate m - 1 times. Set x = 0 in the result of each differentiation. Use the fact that the degree of N(x) is less than m; the degree of M(x) is less than n.

642 Use Lagrange's formula. Perform the division in every term of the result and use the result of Problem 100 to simplify the individual terms.

644 First use Lagrange's interpolation formula to express $f(x_0)$ in terms of $f(x_1)$, $f(x_2)$, $\cdots$, $f(x_n)$. Substitute this in the given equation. Note that the quantities $f(x_1)$, $f(x_2)$, $\cdots$, $f(x_n)$ are independent. Finally use the formula $\varphi(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$, and expand in powers of $x - x_0$.

645 The polynomial x^s is determined by its values, by means of Lagrange's interpolation formula.

648, 649 Use Newton's interpolation formula.

650 Find the values of the polynomial for $x = 0, 1, 2, 3, \cdots, 2n$.

651 The problem can be solved by using Newton's interpolation formula. Another way is to consider the polynomial $F(x) = x\, f(x) - 1$, where $f(x)$ is the given polynomial.

652 Consider the polynomial $(x - a)\, f(x) - 1$.

653 Use Newton's interpolation formula. It is convenient to introduce the factorial in the denominator of every term.

654 Let $f(x)$ be the given polynomial; consider the polynomial $f(x^2)$.

655 If $x_1, x_2, \cdots, x_n$ are the roots of the denominator, then Lagrange's formula reads:

$$\frac{f(x)}{\varphi(x)} = \sum_{k=1}^{n} \frac{f(x_k)}{(x - x_k)\varphi'(x_k)}$$

656 First use the Lagrange methods explained in the Hint above; then select terms that are complex conjugates.

657 e) Use Problem 631. f) Set $\frac{a + x}{2a} = y$.

d), h) Write out the general form of the answer. Multiply through by the greatest common denominator and set $x = x_1, x_2, \cdots, x_n$. Then make the same substitution after differentiating the identity.

660 Use Problem 659. For part b) separate $\frac{1}{x^2 - 3x + 2}$ into partial fractions.

665, 666 Use the result of Problem 663.

667 In part c) expand the polynomial in powers of $x - 1$.

668 Expand in the powers of $x - 1$ [set $x = y + 1$].

669 Set $x = y + 1$; use mathematical induction to prove that all the coefficients of the dividend and divisor (except the coefficient of the highest power) are divisible by p.

670, 671 The proof is similar to the proof of Eisenstein's criterion.

679, 680 Assuming that $f(x)$ is reducible, set $x = a_1, a_2, \cdots, a_n$ and note what values the factors of $f(x)$ would have to have.

681 If proper divisors existed, find how many equal values could occur.

682 Use the fact that $f(x)$ has no real roots.

683 Show that a polynomial that has more than three integral roots cannot take a prime value for an integral value of its argument. Apply this remark to the polynomial $f(x) - 1$.

684, 685 Use the result of Problem 683.

702 Set up a Sturm sequence and consider the cases n odd, n even separately.

707-712 Find recurrence relations among the polynomials of consecutive degrees and their derivatives; construct Sturm sequences from these. In Problem 708 set up a Sturm sequence only for positive values of x and use other arguments to prove the non-existence of negative roots. In Problem 709 set up a Sturm sequence for negative x.

713 Use the fact that $F'(x) = 2f(x)\, f'''(x)$, and that $f'''(x)$ is constant.

717 Use the factored form of $g(x)$; use the results of 716 several times.

718 Apply 717 in the special case $g(x) = x^m$.

719 Use the fact that if all roots of the polynomial

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n \quad \text{are real,}$$

then all roots of the polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_0 \quad \text{are also real.}$$

721 Multiply by $x - 1$.

727 Assume the assertion false; use Rolle's theorem and the result of Problem 581.

728 Consider the graph of $\psi(x) = \frac{f(x)}{f'(x)}$. Show that every root of $[f'(x)]^2 - f(x)f''(x)$ corresponds to an extremum of $\psi(x)$ and conversely. Show that $\psi(x)$ cannot have extrema in an interval bounded by roots of $f'(x)$ that contains a root of $f(x)$; and can have only one extremum in an interval that does not contain a root of $f(x)$.

729 Use the results of Problems 727, 726.

730 Elucidate the behavior of the function

$$\psi(x) = \frac{f(x)}{f'(x)} + \frac{x+\lambda}{\gamma}.$$

731 Set $\lambda = 0$ in the result of the preceding problem.

732 Use induction on the degree of $f(x)$; set $f(x) = (x + \lambda) f_1(x)$, where $f_1(x)$ is a polynomial of degree $n - 1$.

733 Apply the result of preceding problem twice.

734 If all the roots of $f(x)$ are positive, then the proof follows from elementary considerations using induction on the degree of $f(x)$. As an

inductive hypothesis use the statement that the roots $x_1, x_2, \cdots, x_{n-1}$ of the polynomial $b_0 + b_1 wx + \ldots + b_{n-1} w^{(n-1)^2} x^{n-1}$ satisfy the relations

$$0 < x_1 < x_2 < \ldots < x_{n-1} \quad , \quad x_i > x_{i-1} w^{-2}.$$

To prove the theorem in the general case write w^{x^2} as the limit of a polynomial in x with the roots lying exterior to the interval (0, n); and use the result of Problem 731.

735, 736 Consider $\left| \frac{\varphi(x) + i\psi(x)}{\varphi(x) - i\psi(x)} \right|$, where

$$\varphi(x) = a_0 \cos\varphi + \ldots + a_n \cos(\varphi + n\theta)\, x^n,$$
$$\psi(x) = b_0 \sin\varphi + \ldots + b_n \sin(\varphi + n\theta)\, x^n.$$

or

$$\varphi(x) = a_0 + a_1 x + \ldots + a_n x^n,$$
$$\psi(x) = b_0 + b_1 x + \ldots + b_n x^n.$$

In 736, to show the roots are real, multiply $\varphi(x) + i\psi(x)$ by $\alpha - \beta i$ and examine the real part. Use the result of Problem 727.

737 Resolve $\frac{\psi(x)}{\varphi(x)}$ into partial fractions. Note the signs of the coefficients in this resolution, and examine the imaginary part of

$$\frac{-i\,[\varphi(x) + i\psi(x)]}{\varphi(x)} = \frac{\psi(x)}{\varphi(x)} - i.$$

738 Examine the imaginary part of $\frac{f'(x)}{f(x)}$, when it is written as a sum of partial fractions.

739 Note that a simple substitution of the argument will transform the given half plane into the half plane $\operatorname{Im}(x) > 0$; see the preceding problem.

740 Show how to apply Problem 739.

741 Separate $\frac{f'(x)}{f(x)}$ into partial fractions and evaluate the imaginary parts.

743 Set $x = yi$ and use the result of Problems 736, 737.

744, 745 Use the result of Problem 743.

746 Set $x = (1 + y)/(1 - y)$ and use the result of Problem 744.

747 Multiply the polynomial by $1 - x$ and calculate the modulus of $(1 - x)\, f(x)$ for $|x| = \rho > 1$.

CHAPTER VI - HINTS
SYMMETRIC FUNCTIONS

772 If a triangle similar to the given triangle is inscribed in a circle of radius 1/2, its sides will be equal to the sines of the angles of the given triangle.

800, 801 First consider the sum

$$\sum_{i=1}^{n} (x + x_i)^k,$$

then set $x = x_j$ and sum on j from 1 to n. Finally, combine repeated terms and divide by 2.

805 Every primitive n-th root of unity when raised to the m-th power gives a primitive (n/d)-th root, where d = (m,n) is the greatest common divisor of m and n. Moreover if we operate in this way on the primitive n-th roots of unity, we obtain all the primitive (n/d)-th roots of unity without repetition.

806 Use the results of problems 805, 117, 119.

807 Note that the equation that has roots $x_1, x_2, \cdots, x_n$ is the determinantal equation of problem 803; this follows from Newton's formula expressing the coefficients in terms of power sums of the roots.

808 The problem is easily solved by using Newton's formula or by expressing the Newton elementary functions in terms of the fundamental symmetric functions. See the determinant in problem 802. Another method is to multiply the equation by $(x - a) \cdot (x - b)$ and calculate the power sums in the resulting equation.

809 First multiply the equation by $(x - a) \cdot (x - b)$.

818 Think of the roots of the polynomial $f(x)$ as independent variables. Multiply the determinant of coefficients by the Vandermondian.

819, 820 First show that every polynomial ψ_k has degree $n - 1$. Then multiply the determinant of the coefficients of ψ_k by the Vandermondian.

827 Use the fact that the primitive m-th roots of the n-th roots of unity run through all the primitive (n/d)-th roots of unity, where $d = (m,n)$.

828 Use the result of problem 827 and the fact that $R(X_m, X_n)$ is a divisor of $R(X_m, x^n - 1)$ and $R(X_n, x^m - 1)$.

834, 835 Calculate $R(f', f)$.

839 Multiply by $x - 1$.

840 Multiply by $x - 1$ and use the result of Problem 835.

843 Calculate $R(X_n, X'_n)$. To calculate X'_n at the roots of X_n, write X_n in the form

$$(x^n - 1)\Pi(x^d - 1)^{\mu\left(\frac{n}{d}\right)},$$

where d runs through the proper divisors of n.

844 Use the relation $E'_n = E_n - x^n$.

845 Use the relation

$$(nx - x - a)F_n - x(x+1)F'_n + \frac{(a-1)\ldots(a-n)}{n!} = 0.$$

846 Use the relation

$$P_n = xP_{n-1} - (n-1)P_{n-2};\quad P'_n = nP_{n-1}.$$

847 Use the relation

$$xP'_n = nP_n + n^2P_{n-1};\quad P_n = (x - 2n + 1)P_{n-1} - (n-1)^2P_{n-2}.$$

848 Use the relation

$$(4 - x^2)P'_n + nxP_n = 2nP_{n-1};\quad P_n - xP_{n-1} + P_{n-2} = 0.$$

849 Use the relation

$$P_n - 2xP_{n-1} + (x^2+1)P_{n-2} = 0;\quad P'_n = (n+1)P_{n-1}.$$

850 Use the relation

$$P_n - (2n-1)xP_{n-1} + (n-1)^2(x^2+1)P_{n-2} = 0;\quad P'_n = n^2P_{n-1}.$$

851 Use the relation

$$P_n - (2nx+1)P_{n-1} + n(n-1)x^2P_{n-2} = 0;\quad P'_n = (n+1)nP_{n-1}.$$

852 Solve the problem by using the Lagrange polynomial. Note that the maximum in question can be obtained by solving a differential equation; namely setting the derivative equal to 0. Solve this equation by the method of undetermined coefficients.

867 First show that for fixed n, there is only a finite number of equations with the given properties. Then show that the given property is invariant under the substitution $y = x^m$.

CHAPTER VII - HINTS
LINEAR ALGEBRA

884 Use the results of problems 51, 52.

916 The smallest angle must be one of the angles formed by the vectors of the second plane with its orthogonal projection on the first plane.

917 Let the center of the cube be the origin and let its sides be parallel to the coordinate axes. Next take four mutually orthogonal diagonals as axes.

918 Use the result of problem 907.

920 Use induction.

921 Use the fact that
$V[A_1, \ldots, A_m, B_1, \ldots, B_k] = V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k]$,
if $A_i \perp B_j$. Then use the result of the preceding problem.

933 First find the proper values of the square matrix. Then to determine the sign of the square roots use the fact that the sum of the proper values is equal to the sum of the elements on the principal diagonal and that the product of the proper values is equal to the value of the determinant. Use the results of problems 126, 299.

934 Use the result of problem 933.

936 Use the results of problems 537, 930.

943 1) Use the fact that the determinant of a triagonal transformation is unity.

2) Set $x_{k+1} = x_{k+2} = \ldots = x_n = 0.$

945 Take as a new independent variable the linear form of the problem.

946 Use the result of problem 945, by writing f as the sum of the square of a linear form and other terms.

948 Let $x_1, x_2, \cdots, x_n$ be the roots of the given equation. Consider the following quadratic form in the variables $u_1, u_2, \cdots, u_n$:

$$f = \sum_{k=1}^{n} \left(u_1 + u_2 x_k + \ldots + u_n x_k^{n-1}\right)^2 .$$

950 Expand f, φ as a sum of squares and use the distributivity of the operation (f, φ).

965 The converse of the theorem is proved by using the relation X = AX + (E - A) X.

966 Write the matrix of the projection in terms of a basis that is obtained by complementing orthonormal bases of P, Q.

967 Note that (AX, X) = 0 for an arbitrary real vector X. Write a proper value and the corresponding vector as the sum of real and imaginary parts.

968 Let P be an orthogonal matrix, the first two columns of which consist of the normed real and

imaginary parts of a proper vector; form the product $P^{-1}AP$.

970, 971 Use the fact that if X, Y are arbitrary real vectors, and A is an arbitrary orthogonal matrix, then the relation $(AX, AY) = (A'AX, Y) = (X, Y)$ holds.

972 The proof is based on the results of problems 970, 971 in the same way that the proof of 968 is based on the result of problem 967.

975, 976 Transform to Jordan canonical form.

978 This problem is related to the solution of the preceding problem.

980 For the necessity, see problem 473. To establish the sufficiency first consider the case when all the diagonal elements of the matrix C are zero. Then note that if $C = XY - YX$, then the relation

$$S^{-1}CS = (S^{-1}XS)(S^{-1}YS) - (S^{-1}YS)(S^{-1}XS)$$

holds.

981 Try $H = \sum C^k C^{*k}$.

982 Try $H = \sum C^k BC^{*k}$.

983 The matrix $C = (I + A)(I - A)^{-1}$ has all its proper values in the unit circle.

CHAPTER I - SOLUTIONS

COMPLEX NUMBERS

1 $x = -4/11; \quad y = 5/11.$

2 $x = -2; \quad y = 3/2; \quad z = 2; \quad t = -1/2.$

3 1, if $n = 4k$; i, if $n = 4k + 1$; -1, if $n = 4k + 2$; -1, if $n = 4k + 3$; k being an integer.

5 a) $117 + 44i$; b) -556; c) $-76i$.

6 Only in the following cases:

1) If one of the factors is not zero;

2) If the factors have the form $(a + bi)$ or $\lambda(b + ai)$, where λ is real.

7 a) $\cos 2\alpha + i \sin 2\alpha$; b) $\frac{a^2 - b^2}{a^2 + b^2} + i\frac{2ab}{a^2 + b^2}$; c) $\frac{44 - 5i}{318}$; d) $\frac{-1 - 32i}{25}$; e) 2.

8 $2i^{n-1}$.

9 a) $x = 1 + i$, $y = i$; b) $x = 2 + i$, $y = 2 - i$; c) $x = 3 - 11i$, $y = -3 - 9i$, $z = 1 - 7i$.

10 a) $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$; b) 1.

11 a) $a^2 + b^2 + c^2 - (ab + bc + ac)$; b) $a^3 + b^3$; c) $2(a^3 + b^3 + c^3) - 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 12abc$; d) $a^2 - ab + b^2$.

12 a) $0,\ 1,\ -\frac{1}{2}+\frac{i\sqrt{3}}{2},\ -\frac{1}{2}-\frac{i\sqrt{3}}{2}$; b) $0,\ 1,\ i,\ -1,\ -i$.

15 a) $\pm(1+i)$; b) $\pm(2-2i)$; c) $\pm(2-i)$; d) $\pm(1+4i)$; e) $\pm(1-2i)$; f) $\pm(5+6i)$; g) $\pm(1+3i)$; h) $\pm(1-3i)$; i) $\pm(3-i)$; j) $\pm(3+i)$; k) $\pm\left(\sqrt{\frac{\sqrt{13}+2}{2}}-i\sqrt{\frac{\sqrt{13}-2}{2}}\right)$;

l) $\pm\sqrt{8+2\sqrt{17}}$; $\pm i\sqrt{-8+2\sqrt{17}}$; m) $\pm\left(\sqrt{\frac{3}{2}}-i\sqrt{\frac{1}{2}}\right)$;
n) $\frac{\sqrt{2}(\pm 1\pm i)}{2}$; o) $i^{\alpha}\left(\frac{1+\sqrt{3}}{2}+\frac{1-\sqrt{3}}{2}i\right)$, $\alpha=0,\ 1,\ 2,\ 3$.

16 $\pm(\beta-\alpha i)$.

17 a) $x_1=3-i$; $x_2=-1+2i$; b) $x_1=2+i$; $x_2=1-3i$; c) $x_1=1-i$; $x_2=\frac{4-2i}{5}$.

18 a) $1\pm 2i$; $-4\pm 2i$; $(x^2-2x+5)(x^2+8x+20)$;
b) $2\pm i\sqrt{2}$; $-2\pm 2i\sqrt{2}$; $(x^2-4x+6)(x^2+4x+12)$.

19 a) $x=\pm\frac{\sqrt{7}}{2}\pm\frac{i}{2}$; b) $\pm 4\pm i$.

20 $\pm\sqrt{\frac{\sqrt{q}}{2}-\frac{p}{4}}\pm i\sqrt{\frac{\sqrt{q}}{2}+\frac{p}{4}}$.

22 a) $\cos 0+i\sin 0$; b) $\cos\pi+i\sin\pi$; c) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$;
d) $\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$; e) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$;
f) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$; g) $\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$;
h) $\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$; i) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$;
j) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$; k) $2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$;
l) $2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$; m) $2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$;
n) $3(\cos\pi+i\sin\pi)$; o) $2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$;
p) $(\sqrt{2}+\sqrt{6})\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$.

Remark: Here we have given just one of the possible values of the argument.

23 a) $\sqrt{10}\,(\cos 18^\circ 26' + i \sin 18^\circ 26')$;
b) $\sqrt{17}\,(\cos 345^\circ 57' 48'' + i \sin 345^\circ 57' 48'')$;
c) $\sqrt{5}\,(\cos 153^\circ 26' 6'' + i \sin 153^\circ 26' 6'')$;
d) $\sqrt{5}\,(\cos 243^\circ 26' 6'' + i \sin 243^\circ 26' 6'')$.

24 a) A circle of radius 1 with center at the origin.
b) A half ray from the origin making an angle $\pi/6$ with the polar axis.

25 a) The interior of a cicle with center at the origin and radius 2.
b) The interior and circumference of a circle with center at the point (0, 1) and radius 1.
c) The interior of a circle with center at the point (1, 1) and radius 1.

26 a) $x = 3/2 - 2i$; b) $x = 3/4 + i$.

27 The result is equivalent to the following geometric proposition: The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

29 When the difference between the arguments of these numbers is $\pi + 2k\pi$, for some integer k.

30 When the difference in the arguments of these numbers is an integral multiple of 2π.

34 $\cos(\varphi+\psi)+i\sin(\varphi+\psi)$.

35 $\frac{\sqrt{2}}{2}\left[\cos\left(2\varphi-\frac{\pi}{12}\right)+i\sin\left(2\varphi-\frac{\pi}{12}\right)\right]$.

36 a) $2^{12}(1+i)$; b) $2^9(1-i\sqrt{3})$; c) $(2-\sqrt{3})^{12}$; d) -64.

38 $\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}$.

39 $2\cos\frac{2n\pi}{3}$.

40 Solution. $1+\cos\alpha+i\sin\alpha=$

$$=2\cos^2\frac{\alpha}{2}+2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right);$$

$$(1+\cos\alpha+i\sin\alpha)^n=2^n\cos^n\frac{\alpha}{2}\left(\cos\frac{n\alpha}{2}+i\sin\frac{n\alpha}{2}\right).$$

43 a) $-i$; $\frac{\sqrt{3}+i}{2}$; $\frac{-\sqrt{3}+i}{2}$;

b) $-1+i$; $\frac{1+\sqrt{3}}{2}+\frac{\sqrt{3}-1}{2}i$; $\frac{1-\sqrt{3}}{2}-\frac{1+\sqrt{3}}{2}i$;

c) $1+i$; $1-i$; $-1+i$; $-1-i$;

d) 1; -1; $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$; $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$; $\frac{1}{2}+i\frac{\sqrt{3}}{2}$; $\frac{1}{2}-i\frac{\sqrt{3}}{2}$;

e) $i\sqrt{3}$; $-i\sqrt{3}$; $\frac{3+i\sqrt{3}}{2}$; $\frac{3-i\sqrt{3}}{2}$; $-\frac{3+i\sqrt{3}}{2}$; $\frac{-3+i\sqrt{3}}{2}$.

44 a) $\sqrt[6]{5}(\cos 8^\circ 5' 18''+i\sin 8^\circ 5' 18'')\varepsilon_k$,

where $\varepsilon_k=\cos 120^\circ k+i\sin 120^\circ k$, $k=0, 1, 2$;

b) $\sqrt[6]{10}(\cos 113^\circ 51' 20''+i\sin 113^\circ 51' 20'')\varepsilon_k$,

where $\varepsilon_k=\cos 120^\circ k+i\sin 120^\circ k$, $k=0, 1, 2$;

c) $\sqrt[10]{13}(\cos 11^\circ 15' 29''+i\sin 11^\circ 15' 29'')\varepsilon_k$,

where $\varepsilon_k=\cos 72^\circ k+i\sin 72^\circ k$, $k=0, 1, 2, 3, 4$.

45 a) $\frac{1}{\sqrt[12]{2}}\left(\cos\frac{24k+19}{72}\pi + i\sin\frac{24k+19}{72}\pi\right)$,

where k = 0, 1, 2, 3, 4, 5;

b) $\frac{1}{\sqrt[16]{2}}\left(\cos\frac{24k+5}{96}\pi + i\sin\frac{24k+5}{96}\pi\right)$,

where k = 0, 1, 2, 3, 4, 5, 6, 7;

c) $\frac{1}{\sqrt[12]{2}}\left(\cos\frac{24k+17}{72}\pi + i\sin\frac{24k+17}{72}\pi\right)$,

for k = 0, 1, 2, 3, 4, 5.

46 $\beta\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$, for k = 0, 1, 2, $\cdots$, n - 1.

47 a) Solution. Consider $(\cos x + i \sin x)^5$. By DeMoivre's theorem, we have

$$(\cos x + i \sin x)^5 = \cos 5x + i \sin 5x.$$

On the other hand,

$$(\cos x + i\sin x)^5 = \cos^5 x + 5i\cos^4 x \sin x - 10\cos^3 x \sin^2 x - $$
$$- 10i\cos^2 x \sin^3 x + 5\cos x \sin^4 x + i\sin^5 x =$$
$$= (\cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x) + i(5\cos^4 x \sin x -$$
$$- 10\cos^2 x \sin^3 x + \sin^5 x).$$

Collecting these results we obtain:

$$\cos 5x = \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x;$$

b) $\cos^8 x - 28\cos^6 x \sin^2 x + 70\cos^4 x \sin^4 x - 28\cos^2 x \sin^6 x + \sin^8 x$;

c) $6\cos^5 x \sin x - 20\cos^3 x \sin^3 x + 6\cos x \sin^5 x$;

d) $7\cos^6 x \sin x - 35\cos^4 x \sin^3 x + 21\cos^2 x \sin^5 x - \sin^7 x$.

48 $\frac{2(3\,\mathrm{tg}\,\varphi - 10\,\mathrm{tg}^3\varphi + 3\,\mathrm{tg}^5\varphi)}{1 - 15\,\mathrm{tg}^2\varphi + 15\,\mathrm{tg}^4\varphi - \mathrm{tg}^6\varphi}$.

49 $\cos nx = \cos^n x - C_2^n \cos^{n-2} x \sin^2 x + C_4^n \cos^{n-4} x \sin^4 x - \ldots + M,$

where $M = (-1)^{\frac{n}{2}} \sin^n x,$ if n is even;

$M = (-1)^{\frac{n-1}{2}} n \cos x \sin^{n-1} x,$ if n is odd.

$\sin nx = C_1^n \cos^{n-1} x \sin x - C_3^n \cos^{n-3} x \sin^3 x + \ldots + M,$

where $M = (-1)^{\frac{n-2}{2}} n \cos x \sin^{n-1} x,$ n is even;

$M = (-1)^{\frac{n-1}{2}} \sin^n x,$ if n is odd.

50 a) Solution. Let $\alpha = \cos x + i \sin x$. Then

$$\alpha^{-1} = \cos x - i \sin x;$$
$$\alpha^k = \cos kx + i \sin kx; \quad \alpha^{-k} = \cos kx - i \sin kx.$$

Thus we obtain $\cos kx = \frac{\alpha^k + \alpha^{-k}}{2}$; $\sin kx = \frac{\alpha^k - \alpha^{-k}}{2i}$.

In particular

$$\cos x = \frac{\alpha + \alpha^{-1}}{2}; \quad \sin x = \frac{\alpha - \alpha^{-1}}{2i};$$
$$\sin^3 x = \left(\frac{\alpha - \alpha^{-1}}{2i}\right)^3 = \frac{\alpha^3 - 3\alpha + 3\alpha^{-1} - \alpha^{-3}}{-8i} = \frac{(\alpha^3 - \alpha^{-3}) - 3(\alpha - \alpha^{-1})}{-8i};$$
$$\sin^3 x = \frac{2i \sin 3x - 6i \sin x}{-8i} = \frac{3 \sin x - \sin 3x}{4};$$

b) $\frac{\cos 4x - 4 \cos 2x + 3}{8}$; c) $\frac{\cos 5x + 5 \cos 3x + 10 \cos x}{16}$;

d) $\frac{\cos 6x + 6 \cos 4x + 15 \cos 2x + 10}{32}$.

52 Solution.

$$C_p^{m-p} + C_{p-1}^{m-p-1} = \frac{(m-p)(m-p-1)\ \ldots\ (m-2p+1)}{p!} +$$
$$+ \frac{(m-p-1)\ldots(m-2p+1)}{(p-1)!} = \frac{m(m-p-1)(m-p-2)\ldots(m-2p+1)}{p!}.$$

Set $2 \cos mx = S_m$; $2 \cos x = a$. Then the equality we are looking for can be written in the following form:

$$S_m = a^m - ma^{m-2} + (C_2^{m-2} + C_1^{m-3})a^{m-4} - \dots$$
$$\dots + (-1)^p (C_p^{m-p} + C_{p-1}^{m-p-1})a^{m-2p} + \dots$$

It is not difficult to show that:

$$2 \cos mx = 2 \cos x \cdot 2 \cos (m - 1) x - 2 \cos(m - 2)x,$$

or in our notation $S_m = aS_{m-1} - S_{m-2}$.

We first check that for $m = 1$, $m = 2$ the inequality we are trying to prove is valid. Then we use induction. We assume that

$$S_{m-1} = a^{m-1} - (m-1)a^{m-3} + (C_2^{m-3} + C_1^{m-4})a^{m-5} + \dots$$
$$\dots + (-1)^p (C_p^{m-p-1} + C_{p-1}^{m-p-2})a^{m-2p-1} + \dots;$$
$$S_{m-2} = a^{m-2} - (m-2)a^{m-4} + (C_2^{m-4} + C_1^{m-5})a^{m-6} + \dots$$
$$\dots + (-1)^{p-1} (C_{p-1}^{m-p-1} + C_{p-2}^{m-p-2})a^{m-2p} + \dots$$

Then $S_m = a^m - ma^{m-2} + \dots$

$$\dots + (-1)^p (C_p^{m-p-1} + C_{p-1}^{m-p-2} + C_{p-1}^{m-p-1} + C_{p-2}^{m-p-2})a^{m-2p} + \dots$$

Using the fact that $C_k^n = C_k^{n-1} + C_{k-1}^{n-1}$, we establish the desired result.

53 $$\frac{\sin mx}{\sin x} = (2 \cos x)^{m-1} - C_1^{m-2} (2 \cos x)^{m-3} +$$
$$+ C_2^{m-3}\, 2 \cos x)^{m-5} - \dots + (-1)^p C_p^{m-p-1} (2 \cos x)^{m-2p-1} + \dots$$

54 a) $2^{\frac{n}{2}} \cos \frac{n\pi}{4}$; b) $2^{\frac{n}{2}} \sin \frac{n\pi}{4}$.

56 $\frac{2^n}{3^{\frac{n-1}{2}}} \sin \frac{n\pi}{6}$.

59 a) Solution.

$$S = 1 + a\cos\varphi + a^2\cos 2\varphi + \dots + a^k \cos k\varphi.$$

We find $T = a\sin\varphi + a^2\sin 2\varphi + \dots + a^k \sin k\varphi$;

$$S + Ti = 1 + a(\cos\varphi + i\sin\varphi) + a^2(\cos 2\varphi + i\sin 2\varphi) + \dots$$
$$\dots + a^k(\cos k\varphi + i\sin k\varphi)$$

Set $\alpha = \cos\varphi + i\sin\varphi$. Then

$$S + Ti = 1 + a\alpha + a^2\alpha^2 + \dots + a^k\alpha^k = \frac{a^{k+1}\alpha^{k+1} - 1}{a\alpha - 1}.$$

S is the real part of this sum. Moreover,

$$S + Ti = \frac{a^{k+1}\alpha^{k+1} - 1}{a\alpha - 1} \cdot \frac{a\alpha^{-1} - 1}{a\alpha^{-1} - 1} = \frac{a^{k+2}\alpha^k - a^{k+1}\alpha^{k+1} - a\alpha^{-1} + 1}{a^2 - a(\alpha + \alpha^{-1}) + 1}.$$

Thus $S = \frac{a^{k+2}\cos k\varphi - a^{k+1}\cos(k+1)\varphi - a\cos\varphi + 1}{a^2 - 2a\cos\varphi + 1}$.

b) $\frac{a^{k+2}\sin(\varphi + kh) - a^{k+1}\sin[\varphi + (k+1)h] - a\sin(\varphi - h) + \sin\varphi}{a^2 - 2a\cos h + 1}$.

c) $\frac{\sin\frac{2n+1}{2}x}{2\sin\frac{x}{2}}$.

60 Solution.

$$T = \sin x + \sin 2x + \cdots + \sin nx;$$

$$S = \cos x + \cos 2x + \cdots + \cos nx.$$

Let $\alpha = \cos\frac{x}{2} + i\sin\frac{x}{2}$. Then $S + Ti = \alpha^2 + \alpha^4 + \ldots + \alpha^{2n}$,

$$S+Ti=\alpha^2\frac{\alpha^{2n}-1}{\alpha^2-1}=\alpha^2\frac{\alpha^n(\alpha^n-\alpha^{-n})}{\alpha(\alpha-\alpha^{-1})}=\left(\cos\frac{n+1}{2}x+i\sin\frac{n+1}{2}x\right)\frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}.$$

Thus

$$T=\sin\frac{n+1}{2}x\,\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

61 $\dfrac{2(2-\cos x)}{5-4\cos x}$.

64 a) $\dfrac{\sin\left(a+\frac{n-1}{2}h\right)\sin\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is even,

$\dfrac{\cos\left(a+\frac{n-1}{2}h\right)\cos\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is odd;

b) $\dfrac{\cos\left(a+\frac{n-1}{2}h\right)\sin\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is even,

$\dfrac{\sin\left(a+\frac{n-1}{2}h\right)\cos\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is odd.

66 a) $2^n\cos^n\frac{x}{2}\cos\frac{n+2}{2}x$; b) $2^n\cos^n\frac{x}{2}\sin\frac{n+2}{2}x$.

67 a) $2^n\sin^n\frac{x}{2}\cos\frac{n\pi-(n+2)x}{2}$; b) $2^n\sin^n\frac{x}{2}\sin\frac{(n+2)x-n\pi}{2}$.

68 The limit of the sum is equal to the vector corresponding to the number (3 + i)/5.

69 $\frac{n}{2}-\frac{\sin 4nx}{4\sin 2x}$.

71 a) $\dfrac{3\cos\frac{n+1}{2}x\sin\frac{nx}{2}}{4\sin\frac{x}{2}}+\dfrac{\cos\frac{3(n+1)}{2}x\sin\frac{3nx}{2}}{4\sin\frac{3x}{2}}$;

b) $\dfrac{3\sin\frac{n+1}{2}x\sin\frac{nx}{2}}{4\sin\frac{x}{2}}-\dfrac{\sin\frac{3(n+1)}{2}x\sin\frac{3nx}{2}}{4\sin\frac{3x}{2}}$.

72 a) $\dfrac{(n+1)\cos nx-n\cos(n+1)x-1}{4\sin^2\frac{x}{2}}$; b) $\dfrac{(n+1)\sin nx-n\sin(n+1)x}{4\sin^2\frac{x}{2}}$.

73 $e^a(\cos b+i\sin b)$.

75 a) -3; $\frac{3\pm i\sqrt{3}}{2}$; b) -3; $\frac{3\pm 5i\sqrt{3}}{2}$;

c) -7; $-1\pm i\sqrt{3}$; d) -1; $\frac{-5\pm 5i\sqrt{3}}{2}$; e) 2; $-1\pm\sqrt{3}$;

f) $\sqrt[3]{2}-\sqrt[3]{4}$; $\frac{\sqrt[3]{4}-\sqrt[3]{2}}{2}\pm\frac{i\sqrt{3}}{2}(\sqrt[3]{4}+\sqrt[3]{2})$;

g) $\sqrt[3]{9}-2\sqrt[3]{3}$; $\frac{2\sqrt[3]{3}-\sqrt[3]{9}}{2}\pm\frac{i\sqrt{3}}{2}(\sqrt[3]{9}+2\sqrt[3]{3})$;

h) $1-\sqrt[3]{2}-\sqrt[3]{4}$; $\frac{2+\sqrt[3]{2}+\sqrt[3]{4}}{2}\pm\frac{i\sqrt{3}}{2}(\sqrt[3]{4}-\sqrt[3]{2})$;

i) $-(1+\sqrt[3]{3}+\sqrt[3]{9})$; $\frac{-2+\sqrt[3]{3}+\sqrt[3]{9}}{2}\pm\frac{i\sqrt{3}}{2}(\sqrt[3]{9}-\sqrt[3]{3})$;

j) 2; $-1\pm 2i\sqrt{3}$; k) 2; $-1\pm 3i\sqrt{3}$; l) 2; $-1\pm 4i\sqrt{3}$;

m) 1; $-2\pm\sqrt{3}$; n) 4; $-1\pm 4i\sqrt{3}$; o) $-2i$; i; i;

p) $-1-i$; $-1-i$; $2+2i$;

q) $-(a+b)$; $\frac{a+b}{2}\pm\frac{i\sqrt{3}}{2}(a-b)$;

r) $-\left(a\sqrt[3]{f^2g}+b\sqrt[3]{fg^2}\right)$;
$\dfrac{a\sqrt[3]{f^2g}+b\sqrt[3]{fg^2}}{2} \pm \dfrac{i\sqrt{3}}{2}\left(a\sqrt[3]{f^2g}-b\sqrt[3]{fg^2}\right)$;

s) 2.1149; -0.2541; -1.8608;

t) 1.5981; 0.5115; -2.1007.

76 Solution.

$$\begin{aligned}
&x_1-x_2=\alpha(1-\omega)+\beta(1-\omega^2)=(1-\omega)(\alpha-\beta\omega^2)\\
&x_1-x_3=\alpha(1-\omega^2)+\beta(1-\omega)=(1-\omega^2)(\alpha-\beta\omega)\\
&x_2-x_3=\alpha(\omega-\omega^2)+\beta(\omega^2-\omega)=(\omega-\omega^2)(\alpha-\beta)\\
&(x_1-x_2)(x_1-x_3)(x_2-x_3)=3(\omega-\omega^2)(\alpha^3-\beta^3);\\
&(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2=\\
&\qquad=-27\left[(\alpha^3+\beta^3)^2-4\alpha^3\beta^3\right]=-27q^2-4p^3.
\end{aligned}$$

77 Solution.

The cubic equation mentioned in the Hints is $z^3 - 3(px + q)z + x^3 + p^3 - 3qx - 3pq = 0$, which has the obvious root $z = -(x + p)$. The two other roots of the cubic equation are

$z_{2,3}=\dfrac{x+p\pm\sqrt{-3(x-p)^2+12q}}{2}$. From problem 76, part 1, the desired equation can be written in the form

$$\begin{aligned}
&-\frac{1}{27}(z_2-z_3)^2(z_3-z_1)^2(z_1-z_2)^2=-\frac{1}{27}\left[-3(x-p)^2+12q\right]\\
&\times\left[\frac{3(x+p)+\sqrt{-3(x-p)^2+12q}}{2}\right]^2\left[\frac{3(x+p)-\sqrt{-3(x-p)^2+12q}}{2}\right]^2=\\
&\qquad=\left[(x-p)^2-4q\right](x^2+px+p^2-q)^2,
\end{aligned}$$

the roots of which are easily seen to be:

$$x_{1,2} = p \pm 2\sqrt{q}; \quad x_3 = x_4 = \frac{-p + \sqrt{4q - 3p^2}}{2};$$
$$x_5 = x_6 = \frac{-p - \sqrt{4q - 3p^2}}{2}.$$

78 The left member can be written in the form

$$\alpha^5 + \beta^5 + 5(\alpha + \beta)(\alpha^2 + \alpha\beta + \beta^2 - a)(\alpha\beta - a) - 2b = 0.$$

Answer. $x = \alpha + \beta$, where

$$\alpha = \sqrt[5]{b + \sqrt{b^2 - a^5}}; \quad \beta = \sqrt[5]{b - \sqrt{b^2 - a^5}}; \quad \alpha\beta = a.$$

79 a) $\pm\sqrt{2}$; $1 \pm i\sqrt{3}$; b) $-1 \pm \sqrt{6}$; $\pm i\sqrt{3}$;

c) $\pm\sqrt{2}$; $\frac{1 \pm i\sqrt{3}}{2}$; d) $\frac{1 \pm \sqrt{5}}{2}$; $\frac{3 \pm \sqrt{5}}{2}$;

e) $\frac{1 \pm \sqrt{13}}{2}$; $1 \pm i$; f) $\frac{1 \pm \sqrt{29}}{2}$; $\frac{5 \pm i\sqrt{7}}{2}$;

g) $\pm i$; $1 \pm i\sqrt{2}$; h) $\pm\sqrt{5}$; $\frac{1 \pm i\sqrt{7}}{2}$;

i) $\pm i$; $-1 \pm i\sqrt{6}$; j) $-2 \pm 2\sqrt{2}$; $-1 \pm i$;

k) 1; 3; $1 \pm \sqrt{2}$; l) 1; -1; $1 \pm 2i$;

m) $\frac{1 + \sqrt{5} \pm \sqrt{22 + 2\sqrt{5}}}{4}$; $\frac{1 - \sqrt{5} \pm \sqrt{22 - 2\sqrt{5}}}{4}$;

n) $\frac{1 + \sqrt{5} \pm \sqrt{30 - 6\sqrt{5}}}{4}$; $\frac{1 - \sqrt{5} \pm \sqrt{30 + 6\sqrt{5}}}{4}$;

o) $\frac{1 + \sqrt{2}}{2} \pm \frac{1}{2}\sqrt{-1 - 2\sqrt{2}}$; $\frac{1 - \sqrt{2}}{2} \pm \frac{1}{2}\sqrt{-1 + 2\sqrt{2}}$;

p) $1 + \sqrt{7} \pm \sqrt{6 + 2\sqrt{7}}$; $1 - \sqrt{7} \pm \sqrt{6 - 2\sqrt{7}}$;

q) $\frac{1 \pm \sqrt{4\sqrt{3} - 3}}{2}$; $\frac{1 \pm \sqrt{-4\sqrt{3} - 3}}{2}$;

r) $\frac{1 + \sqrt{5} \pm \sqrt{-2 - 6\sqrt{5}}}{4}$; $\frac{1 - \sqrt{5} \pm \sqrt{-2 + 6\sqrt{5}}}{4}$;

s) $\frac{1 + \sqrt{2} \pm \sqrt{-5 + 2\sqrt{2}}}{4}$; $\frac{1 - \sqrt{2} \pm \sqrt{-5 - 2\sqrt{2}}}{4}$;

t) $\frac{1 + \sqrt{3} \pm \sqrt{12 + 2\sqrt{3}}}{4}$; $\frac{1 - \sqrt{3} \pm \sqrt{12 - 2\sqrt{3}}}{4}$.

80 Solution.

$$x^4+ax^3+bx^2+cx+d=$$
$$=\left(x^2+\frac{a}{2}x+\frac{\lambda}{2}+mx+n\right)\left(x^2+\frac{a}{2}x+\frac{\lambda}{2}-mx-n\right);$$
$$x_1x_2=\frac{\lambda}{2}+n;\quad x_3x_4=\frac{\lambda}{2}-n;\quad \lambda=x_1x_2+x_3x_4.$$

81 a) ± 1; b) 1; $-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; c) ± 1; $\pm i$;

d) ± 1; $\pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; e) ± 1; $\pm i$; $\pm\frac{\sqrt{2}}{2}(1\pm i)$;

f) ± 1; $\pm i$; $\pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; $\pm\frac{\sqrt{3}}{2}\pm\frac{i}{2}$;

g) ± 1; $\pm i$; $\pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; $\pm\frac{\sqrt{2}}{2}(1\pm i)$; $\pm\frac{\sqrt{3}}{2}\pm\frac{i}{2}$;

$$\pm\frac{\sqrt{6}+\sqrt{2}}{4}\pm i\frac{\sqrt{6}-\sqrt{2}}{4};\quad \pm\frac{\sqrt{6}-\sqrt{2}}{4}\pm i\frac{\sqrt{6}+\sqrt{2}}{4}.$$

82 a) -1; b) $-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; c) $\pm i$; d) $\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$;

e) $\pm\frac{\sqrt{2}}{2}(1\pm i)$; f) $\pm\frac{\sqrt{3}}{2}\pm\frac{i}{2}$;

g) $\pm\frac{\sqrt{6}+\sqrt{2}}{4}\pm i\frac{\sqrt{6}-\sqrt{2}}{4}$; $\pm\frac{\sqrt{6}-\sqrt{2}}{4}\pm i\frac{\sqrt{6}+\sqrt{2}}{4}$.

83 a) 20; 20; 180; b) 72; 144; 12.

84 $\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}$, for k = 1, 2, 3, 4, 5, 6.

85 a) Set $\epsilon_k=\cos\frac{2k\pi}{16}+i\sin\frac{2k\pi}{16}$. Then we find:

ϵ_0 has index 1;

ϵ_8 has index 2;

ϵ_4, ϵ_{12} have index 4;

ϵ_2, ϵ_6, ϵ_{10}, ϵ_{14} have index 8;

the remaining 16-th roots of unity, namely

$\epsilon_1, \epsilon_3, \epsilon_5, \epsilon_7, \epsilon_9, \epsilon_{11}, \epsilon_{13}, \epsilon_{15}$ are primitive.

b) Set $\varepsilon_k = \cos\frac{2k\pi}{20} + i\sin\frac{2k\pi}{20}$. Then we find:

ϵ_0 has index 1;

ϵ_{10} has index 2;

$\epsilon_5, \epsilon_{15}$ have index 4;

$\epsilon_4, \epsilon_8, \epsilon_{12}, \epsilon_{16}$ have index 5;

$\epsilon_2, \epsilon_6, \epsilon_{14}, \epsilon_{18}$ have index 10;

the remaining 20-th roots of unity, namely

$\epsilon_1, \epsilon_3, \epsilon_7, \epsilon_9, \epsilon_{11}, \epsilon_{13}, \epsilon_{17}, \epsilon_{19}$ are primitive.

c) Set $\varepsilon_k = \cos\frac{2k\pi}{24} + i\sin\frac{2k\pi}{24}$. Then we find:

ϵ_0 has index 1;

ϵ_{12} has index 2;

$\epsilon_8, \epsilon_{16}$ have index 3;

$\epsilon_6, \epsilon_{18}$ have index 4;

$\epsilon_4, \epsilon_{20}$ have index 6;

$\epsilon_3, \epsilon_9, \epsilon_{15}, \epsilon_{21}$ have index 8;

$\epsilon_2, \epsilon_{10}, \epsilon_{14}, \epsilon_{22}$ have index 12;

the remaining 24-th roots of unity, namely

$\epsilon_1, \epsilon_5, \epsilon_7, \epsilon_{11}, \epsilon_{13}, \epsilon_{17}, \epsilon_{19}, \epsilon_{23}$ are primitive.

86 a) $X_1(x) = x - 1$; b) $X_2(x) = x + 1$;
c) $X_3(x) = x^2 + x + 1$; d) $X_4(x) = x^2 + 1$;
e) $X_5(x) = x^4 + x^3 + x^2 + x + 1$; f) $X_6(x) = x^2 - x + 1$;
g) $X_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$; h) $X_8(x) = x^4 + 1$;
i) $X_9(x) = x^6 + x^3 + 1$; j) $X_{10}(x) = x^4 - x^3 + x^2 - x + 1$;
k) $X_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$; l) $X_{12}(x) = x^4 - x^2 + 1$;

m) $X_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$;

n) $X_{105}(x) = x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40}$
$- x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} - x^{28} - x^{26}$
$- x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12}$
$- x^9 - x^8 - 2x^7 - x^6 - x^5 + x^2 + x + 1$.

87 $2/(1 - \epsilon)$.

88 0, if $n > 1$.

89 n, if k is divisible by n; 0, if k is not divisible by n.

90 $m(x^m + 1)$.

91 $-n/(1 - \epsilon)$, if $\epsilon \neq 1$; $n(n + 1)/2$, if $\epsilon = 1$.

92 $-\dfrac{n^2(1-\varepsilon)+2n}{(1-\varepsilon)^2}$, if $\epsilon \neq 1$; $\dfrac{n(n+1)(2n+1)}{6}$, if $\epsilon = 1$.

93 a) $-n/2$; b) $-\frac{1}{2} n \cot(\pi/n)$.

94 a) 1; b) 0; c) -1.

95 $x_0 = 1$;

$$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} + \frac{i}{4}\sqrt{10+2\sqrt{5}};$$
$$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4} + \frac{i}{4}\sqrt{10-2\sqrt{5}};$$
$$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = -\frac{\sqrt{5}+1}{4} - \frac{i}{4}\sqrt{10-2\sqrt{5}};$$
$$x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \frac{\sqrt{5}-1}{4} - \frac{i}{4}\sqrt{10+2\sqrt{5}}.$$

96 $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$; $\cos 18^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$.

97 Solution. Divide the equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ by x^3. Using easy transformations, this leads to the equation

$$\left(x+\frac{1}{x}\right)^3+\left(x+\frac{1}{x}\right)^2-2\left(x+\frac{1}{x}\right)-1=0.$$

Now the equation $z^3 + z^2 - 2z - 1 = 0$ is satisfied by $z=2\cos\frac{4\pi}{7}=-2\sin\frac{\pi}{14}$. Moreover $t=2\sin\frac{\pi}{14}$ satisfies the equation $t^3 - t^2 - 2t + 1 = 0$. These equations are the simplest possible ones in the sense that every other equation that has rational coefficients and has a root in common with these must be of higher degree. The proof of this assertion is most conveniently carried through by advanced methods.

98 Solution. Set $n = 2m$; the equation $x^n - 1 = 0$ has the two real roots 1, -1; and $2m - 2$ complex roots. Indeed $\varepsilon_k=\cos\frac{2k\pi}{2m}+i\sin\frac{2k\pi}{2m}$ is conjugate to $\varepsilon_{2m-k}=\cos\frac{2(2m-k)\pi}{2m}+i\sin\frac{2(2m-k)\pi}{2m}$. Thus we obtain

$$x^{2m}-1=(x^2-1)(x-\varepsilon_1)(x-\bar{\varepsilon}_1)(x-\varepsilon_2)(x-\bar{\varepsilon}_2)\dots$$
$$\dots(x-\varepsilon_{m-1})(x-\bar{\varepsilon}_{m-1});\quad x^{2m}-1$$
$$=(x^2-1)[x^2-(\varepsilon_1+\bar{\varepsilon}_1)x+1]\dots[x^2-(\varepsilon_{m-1}+\bar{\varepsilon}_{m-1})x+1];$$
$$x^{2m}-1=(x^2-1)\prod_{k=1}^{m-1}\left(x^2-2x\cos\frac{k\pi}{m}+1\right).$$

If $n = 2m + 1$, then we obtain

$$x^{2m+1}-1=(x-1)\prod_{k=1}^{m}\left(x^2-2x\cos\frac{2k\pi}{2m+1}+1\right).$$

by using similar reasoning.

99 Solution.

a) We have

$$\frac{x^{2m}-1}{x^2-1}=\prod_{k=1}^{m-1}\left(x^2-2x\cos\frac{k\pi}{m}+1\right).$$

Setting x = 1, we find $m=2^{m-1}\prod_{k=1}^{m-1}\left(1-\cos\frac{k\pi}{m}\right)$,

or, $m=2^{2(m-1)}\prod_{k=1}^{m-1}\sin^2\frac{k\pi}{2m}$, . Finally

$$\frac{\sqrt{m}}{2^{m-1}}=\prod_{k=1}^{m-1}\sin\frac{k\pi}{2m}.$$

The proof of formula b) is similar.

100 Solution. Set x = -a/b in the identity

$$x^n-1=\prod_{k=0}^{n-1}(x-\varepsilon_k)\quad,\quad \varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$$

We find $(-1)^n\frac{a^n}{b^n}-1=(-1)^n\prod_{k=0}^{n-1}\left(\frac{a}{b}+\varepsilon_k\right)$, etc.

101 We start with the obvious factorization

$$\cos n\theta+i\sin n\theta-1=\prod_{k=0}^{n-1}(\cos\theta+i\sin\theta-\varepsilon_k),$$

$$\cos n\theta-i\sin n\theta-1=\prod_{k=0}^{n-1}(\cos\theta-i\sin\theta-\varepsilon_k).$$

Finally we obtain the required result by multiplying these two relations.

102 Solution.

$$\prod_{k=0}^{n-1}\frac{(t+\varepsilon_k)^n-1}{t}=\frac{1}{t^n}\prod_{k=0}^{n-1}\prod_{s=0}^{n-1}(t+\varepsilon_k-\varepsilon_s)=$$
$$=\frac{1}{t^n}\prod_{k=0}^{n-1}\prod_{s=0}^{n-1}\left[t-\varepsilon_k\left(\frac{\varepsilon_s}{\varepsilon_k}-1\right)\right]=\frac{1}{t^n}\prod_{k=0}^{n-1}\prod_{s=0}^{n-1}[t-\varepsilon_k(\varepsilon_s-1)]=$$
$$=\frac{1}{t^n}\prod_{s=0}^{n-1}\prod_{k=0}^{n-1}[t-\varepsilon_k(\varepsilon_s-1)]=\frac{1}{t^n}\prod_{s=0}^{n-1}[t^n-(\varepsilon_s-1)^n]=$$
$$=\prod_{k=1}^{n-1}[t^n-(\varepsilon_k-1)^n].$$

103 From the relation $|x|=|x|^{n-1}$, it follows that $|x| = 0$ or $|x| = 1$. If $|x| = 0$, then $x = 0$. If $|x| = 1$, then $x\bar{x} = 1$.

On the other hand, $x\bar{x} = x^n$. Therefore $x^n = 1$. Thus $x = 0$ or $x = \cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$, $k = 0, 1, 2,\cdots,n - 1$. The converse proposition is easy.

104 Solution. If z satisfies the given equation then $\left|\frac{z-a}{z-b}\right|=\sqrt[n]{\left|\frac{\mu}{\lambda}\right|}$. The geometric locus of points, the distances of which to two given points lie in a given ratio, is a circle (in special cases a straight line).

105 a) We have $\frac{x+1}{x-1}=\varepsilon_k$, where $\varepsilon_k=\cos\frac{2k\pi}{m}+i\sin\frac{2k\pi}{m}$, $k=1, 2, \ldots, m-1$. Moreover $x=\frac{\varepsilon_k+1}{\varepsilon_k-1}$. This last relation can be transformed into the form $x_k=i\operatorname{ctg}\frac{k\pi}{m}$, $k = 1, 2,\cdots,m - 1$;

b) $x_k = \operatorname{ctg}\frac{k\pi}{m}$, $k = 1, 2, \ldots, m-1$;

c) $x_k = \dfrac{a}{\varepsilon_k \sqrt[n]{2} - 1}$,

where $\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $k = 0, 1, 2, \cdots, n-1$.

106 Solution. Set $A = \cos\varphi + i\sin\varphi$. Then $\frac{1+ix}{1-ix} = \eta_k^2$,
where $\eta_k = \cos\frac{\varphi + 2k\pi}{2m} + i\sin\frac{\varphi + 2k\pi}{2m}$, $k = 0, 1, \ldots, m-1$.

Thus $x = \dfrac{\eta_k^2 - 1}{i(\eta_k^2 + 1)} = \dfrac{\eta_k - \eta_k^{-1}}{i(\eta_k + \eta_k^{-1})} = \operatorname{tg}\dfrac{\varphi + 2k\pi}{2m}$.

107 Solution. Proceeding as suggested in the Hints we obtain

$$S + Ti = \mu(1 + \lambda x)^n, \quad S - Ti = \bar{\mu}(1 + \bar{\lambda}x)^n,$$

where $\lambda = \cos\alpha + i\sin\alpha$, $\mu = \cos\varphi + i\sin\varphi$. Thus

$$2S = \mu(1 + \lambda x)^n + \bar{\mu}(1 + \bar{\lambda}x)^n.$$

The equation now takes the form

$\mu(1 + \lambda x)^n + \bar{\mu}(1 + \bar{\lambda}x)^n = 0$;

$$x_k = -\frac{\sin\dfrac{(2k+1)\pi - 2\varphi}{2n}}{\sin\dfrac{(2k+1)\pi - 2\varphi - 2n\alpha}{2n}}; \quad k = 0, 1, 2, \ldots, n-1.$$

108 Solution. Set $\alpha^a = 1$; $\beta^b = 1$. Then
$(\alpha\beta)^{ab} = (\alpha^a)^b \cdot (\beta^b)^a = 1$.

109 Solution. Let ε be a common zero of the polynomials $x^a - 1$, $x^b - 1$; let s be the exponent to which ε belongs. Then s is a common divisor of a, b. Thus s must be 1, and $\varepsilon = 1$. The converse is obvious.

110 Solution. Let α_k, β_s be the a-th and b-th roots of 1; $k = 0, 1, 2, \cdots, a-1$; $s = 0, 1, 2, \cdots, b-1$. The result of problem 108 shows that it will be sufficient to establish the fact that all the products $\alpha_k \beta_s$ are distinct. Suppose on the contrary that $\alpha_{k_1}\beta_{s_1} = \alpha_{k_2}\beta_{s_2}$. Then $\frac{\alpha_{k_1}}{\alpha_{k_2}} = \frac{\beta_{s_2}}{\beta_{s_1}}$, i.e. $\alpha_i = \beta_j$. Now use problem 109 and conclude that $\alpha_i = \beta_j = 1$, i.e. $k_1 = k_2$; $s_1 = s_2$.

111 Solution. Let α, β be primitive a-th and b-th roots of 1. Set $(\alpha\beta)^s = 1$. Then $\alpha^{bs} = 1$; $\beta^{as} = 1$. This shows that bs is divisible by a, and as is divisible by b. Therefore s is divisible by ab.

Let λ be a primitive ab-th root of 1. Then $\lambda = \alpha^k \beta^s$ by problem 110. Suppose the exponent to which α^k belongs is $a_1 < a$. Then $\lambda^{a_1 b} = (\alpha^k)^{a_1 b}(\beta^s)^{a_1 b} = 1$, a contradiction. In the same way it can be shown that β^s is a primitive b-th root of 1.

112 Follows immediately from problem 111.

113 As suggested in the Hints, count all numbers not exceeding p^α that are multiples of p. These are $1 \cdot p,\ 2 \cdot p,\ 3 \cdot p,\ \ldots,\ p^{\alpha-1} \cdot p$. The number of these is clearly $p^{\alpha-1}$. Thus $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha\left(1 - \frac{1}{p}\right)$. From problem 112,

$$\varphi(n) = \varphi(p_1^{\alpha_1})\varphi(p_2^{\alpha_2}) \cdots \varphi(p_k^{\alpha_k}) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

114 Solution. If ε is a primitive n-th root of 1, then the conjugate complex number $\bar{\varepsilon}$ is also a primitive n-th root of 1. Furthermore, since $n > 2$, $\varepsilon \neq \pm 1$.

115 $X_p(x) = x^{p-1} + x^{p-2} + \dots + x + 1.$

116 $X_{p^m}(x) = x^{(p-1)p^{m-1}} + x^{(p-2)p^{m-1}} + \dots + x^{p^{m-1}} + 1.$

117 The way to carry out the suggestion in the Hints is to base the arguments on problem 111.

Let $\alpha_1, \alpha_2, \dots, \alpha_{\varphi(n)}$ be the primitive n-th roots of 1. Then $-\alpha_1, -\alpha_2, \dots, -\alpha_{\varphi(n)}$ are primitive 2n-th roots of 1. Thus

$$X_{2n}(x) = (x + \alpha_1)(x + \alpha_2) \dots (x + \alpha_{\varphi(n)}) =$$
$$= (-1)^{\varphi(n)}(-x - \alpha_1) \dots (-x - \alpha_{\varphi(n)}),$$

or, by problem 114, $X_{2n}(x) = X_n(-x)$.

118 Solution. Let the primitive nd-th roots of 1 be $\varepsilon_k = \cos\frac{2k\pi}{nd} + i\sin\frac{2k\pi}{nd}$, that is, let k, n be relatively prime. By the Euclidean algorithm, we can write $k = nq + r$, $0 < r < n$. Thus

$$\varepsilon_k = \cos\frac{2q\pi + \frac{2r\pi}{n}}{d} + i\sin\frac{2q\pi + \frac{2r\pi}{n}}{d},$$

This means that ε_k is one of the n-th roots of $\eta_r = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$, where η_r is a primitive n-th root of 1; indeed every common factor of r, n is a common factor of k, n.

To establish the converse, let $\eta_r = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$ be a primitive n-th root of 1, so that r, n are relatively prime.

The numbers $\varepsilon_q = \cos\frac{2q\pi + \frac{2r\pi}{n}}{d} + i\sin\frac{2q\pi + \frac{2r\pi}{n}}{d} =$

$$\cos\frac{2\pi(r+nq)}{nd} + i\sin\frac{2\pi(r+nq)}{nd}, \quad q = 0, 1, 2, \ldots, d-1$$

are primitive nd-th roots of 1. For if the numbers r + mq, nd had a common prime divisor p, then $p|n$, $p|r$, a contradiction.

119 Solution. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\varphi(n')}$ be primitive n'-th roots of 1. Then $X_{n'}(x^{n''}) = \prod_{k=1}^{\varphi(n')}(x^{n''} - \varepsilon_k)$.

If we now factor $x^{n''} - \varepsilon_k$:

$$x^{n''} - \varepsilon_k = (x - \varepsilon_{k,1})(x - \varepsilon_{k,2})\cdots(x - \varepsilon_{k,n''})$$

we can write

$$X_n'(x^{n''}) = \prod_{\substack{k=1\\ i=1}}^{\substack{k=\varphi(n')\\ i=n''}} (x - \varepsilon_{k,i}).$$

The result of problem 118 shows that every factor $x - \varepsilon_{k,i}$ is a divisor of $X_n(x)$ and conversely. Thus the degrees of $X_n(x)$, $X_{n'}(x^{n''})$ are equal, for

$$\varphi(n) = n''\varphi(n').$$

121 Solution. The sum of all the n-th roots of 1 is zero. But every n-th root of 1 belongs to an exponent d that divides n, and conversely. Thus

$$\sum_{d/n}\mu(d) = 0.$$

122 Solution. If $\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ belongs to the exponent n_1, the factor $x - \varepsilon_k$ is a divisor of the binomials $x^d - 1$ for which $d|n_1$, and not for those for which $d\nmid n_1$. Moreover if d runs through all

divisors of n divisible by n_1, then n/d runs through all divisors of n/n_1. Therefore $x - \epsilon_k$ appears with exponent $\sum_{d/\frac{n}{n_1}} \mu(d_1)$ in the right member. This sum has the value 0 if $n/n_1 \neq 1$; it has the value 1 if $n = n_1$.

123 Solution. If $n=p^\alpha$ for p prime, then $X_n(1) = p$. If $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, for $p_1, p_2, \cdots, p_k$ distinct primes, then by problem 119, $X_n(1)=X_{n'}(1)$ where $n'=p_1p_2\dots p_k$.

Now set $n = p_1p_2 \cdots p_k$; $k \geq 2$; $n_1 = n/p_k$. Note that one can write down all the divisors of n by multiplying the distinct divisors of n_1 and their products by p_k. Therefore

$$X_n(x)=\prod_{d/n}(x^d-1)^{\mu\left(\frac{n}{d}\right)}=\prod_{d/n_1}(x^d-1)^{\mu\left(\frac{n}{d}\right)}\cdot\prod_{d/n_1}\left(x^{dp_k}-1\right)^{\mu\left(\frac{n}{dp_k}\right)}$$
$$=[X_{n_1}(x)]^{-1}\cdot X_{n_1}\left(x^{p_k}\right).$$

Thus $X_n(x) = 1$.

124 Solution. 1) If n is odd and greater than 1, problem 117 shows that $X_n(-1)=X_{2n}(1)=1$.

2) If $n = 2^k$, then $X_n=\dfrac{x^n-1}{x^{\frac{n}{2}}-1}=x^{\frac{n}{2}}+1$.

But $X_n(-1) = 0$ if $k = 1$; $X_n(-1) = 2$ if $k > 1$.

3) Let $n_1 > 1$ be odd; $n = 2n_1$. By problem 117, $X_n(-1) = X_{n_1}(1)$ and therefore $X_n(-1)$ will have the value p if $n_1 = p^\alpha$ (p prime), or will have the value 1 if $n_1 \neq p^\alpha$.

4) Let $n = 2^k n_1$, k > 1, $n_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, where p_1, p_2, $\cdots$, p_s are distinct odd primes. By problem 119, $X_n(x) = X_{2p_1p_2\dots p_s}(x^\lambda)$, where $\lambda = 2^{k-1} p_1^{\alpha_1 - 1} \dots p_s^{\alpha_s - 1}$. Thus it follows that $X_n(-1) = X_n(1) = 1$.

125 Solution. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{\varphi(n)}$ be the primitive n-th roots of 1. Then:

$$s = \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \dots + \varepsilon_{\varphi(n)-1} \cdot \varepsilon_{\varphi(n)} = \frac{[\mu(n)]^2 - \left(\varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_{\varphi(n)}^2\right)}{2}.$$

1) If n is odd, then ε_i^2 is a primitive n-th root of 1, and $\varepsilon_i^2 = \varepsilon_j^2$ only for i = j. Therefore

$$\varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_{\varphi(n)}^2 = \mu(n) \quad , \quad s = \frac{[\mu(n)]^2 - \mu(n)}{2}.$$

2) Let n_1 be odd, $n = 2n_1$. By problem 111, ε_i is a primitive n_1-th root of 1 and therefore by the paragraph above, $\varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_{\varphi(n)}^2 = \mu(n_1) = -\mu(n)$. Thus, $s = \frac{[\mu(n)]^2 + \mu(n)}{2}$.

3) Let n_1 be odd, k > 1, $n = 2^k n_1$. Then ε_i^2 belongs to the exponent n/2. By problem 118 it follows that $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{\varphi(n)}$ are the square roots of $\eta_1, \eta_2, \dots, \eta_{\varphi\left(\frac{n}{2}\right)}$, and $\eta_1, \eta_2, \dots, \eta_{\varphi\left(\frac{n}{2}\right)}$ are primitive n/2-th roots of 1. Therefore,

$$\varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_{\varphi(n)}^2 = 2\left(\eta_1 + \eta_2 + \dots + \eta_{\varphi\left(\frac{n}{2}\right)}\right) = 2\mu\left(\frac{n}{2}\right); \quad s = -\mu\left(\frac{n}{2}\right).$$

126 Solution. $$S=\sum_{x=0}^{n-1}\varepsilon^{x^2}=\sum_{x=y}^{y+n-1}\varepsilon^{x^2}=\sum_{s=0}^{n-1}\varepsilon^{(y+s)^2}$$ for any integer y;

$$S'=\sum_{y=0}^{n-1}\varepsilon^{-y^2};\quad S'S=\sum_{y=0}^{n-1}\varepsilon^{-y^2}S=\sum_{y=0}^{n-1}\left(\varepsilon^{-y^2}\cdot\sum_{s=0}^{n-1}\varepsilon^{(y+s)^2}\right)=$$
$$=\sum_{y=0}^{n-1}\sum_{s=0}^{n-1}\varepsilon^{2ys+s^2}=\sum_{s=0}^{n-1}\left(\varepsilon^{s^2}\cdot\sum_{y=0}^{n-1}\varepsilon^{2ys}\right)=n+\sum_{s=1}^{n-1}\varepsilon^{s^2}\cdot\sum_{y=0}^{n-1}(\varepsilon^{2s})^y=n$$

for n odd;
$$SS'=n+n\varepsilon^{\left(\frac{n}{2}\right)^2}=n\left[1+(-1)^{\frac{n}{2}}\right]$$

for n even. (Since $\sum_{y=0}^{n-1}\varepsilon^{2sy}=0$ for $n\nmid 2s$.)

Thus, $|S|=\sqrt{n}$, if n is odd;
$$|S|=\sqrt{n\left[1+(-1)^{\frac{n}{2}}\right]},$$
if n is even.

CHAPTER II - SOLUTIONS

COMPUTATION OF DETERMINANTS

127 a) 5; b) 5; c) 1; d) $ab-c^2-d^2$; e) $\alpha^2+\beta^2-\gamma^2-\delta^2$; f) $\sin(\alpha-\beta)$; g) $\cos(\alpha+\beta)$; h) $\sec^2\alpha$; i) -2; j) 0; k) $(b-c)(d-a)$; l) $4ab$; m) -1; n) -1; o) $\frac{1+i\sqrt{3}}{2}$.

128 a) 1; b) 2; c) $2a^2(a+x)$; d) 1; e) -2; f) $-2-\sqrt{2}$; g) $-3i\sqrt{3}$; h) -3.

129 The number of transpositions is odd.

130 a) 10; b) 18; c) 36.

131 a) $i = 8$; $k = 3$; b) $i = 3$; $k = 6$.

132 C_2^n

133 $C_2^n - I$.

134 a) $\frac{n(n-1)}{2}$; b) $\frac{n(n+1)}{2}$.

135 a) $\frac{n(3n+1)}{2}$; b) $\frac{3n(n-1)}{2}$.

136 This is easily proved by considering a pair of elements. Without loss of generality consider the first two a_1, a_2. If these are in their natural order then the permutation in the inverse direction keeps the number 1, 2 in their natural order. Similarly for the pair of numbers a_i, a_k.

On the other hand, if the numbers a_1, a_2 are not in their natural order then the permutation going in the opposite direction involves an inversion to reach the order 1, 2. Similarly for the numbers a_i, a_k.

137 The word algorithm can be obtained from the word logarithm by an even number (4) of transpositions. Hence the permutations are of the same character. Both permutations are odd.

138 a) + ; b) + .

139 a) no; b) yes.

140 $i = 1$; $k = 4$.

141 $a_{11}a_{23}a_{32}a_{44}$, $a_{12}a_{23}a_{34}a_{41}$, $a_{14}a_{23}a_{31}a_{42}$.

142 $-a_{14}a_{32} \det \begin{bmatrix} a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \\ a_{51} & a_{52} & a_{55} \end{bmatrix}$.

143 + .

144 $(-1)^e$, $e = C_2^n$.

146 2; -1.

147 a) $n!$; b) $(-1)^{\frac{n(n-1)}{2}}$; c) $n!$

148 a) $(-1)^{\frac{n(n+1)}{2}}(n!)^{n+1}$; b) $(-1)^{\frac{n(n+1)}{2}}(n!)^{n+1}$.

149 Solution. Write each row as a column. On the one hand, the value of the determinant is unchanged; on the other hand, it is replaced by its complex conjugate.

150 Solution. Write each row as a column. On the one hand, the value of the determinant is unchanged; on the other hand, it is multiplied by -1.

151 $(-1)^{n-1}\Delta$.

152 Multiply by $(-1)^{\frac{n(n-1)}{2}}$.

153 0, since the number of even permutations on n symbols is the same as the number of odd permutations.

154 a) $x_1 = a_1;\ x_2 = a_2;\ \ldots;\ x_{n-1} = a_{n-1};$
b) $x_1 = 0;\ x_2 = 1;\ \ldots;\ x_{n-1} = n-2.$
c) $x_1 = a_1;\ x_2 = a_2;\ \ldots;\ x_{n-1} = a_{n-1}.$

156 0.

158 $(mq - np)\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

159 a) $a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right);$
b) $(-1)^{\frac{n(n-1)}{2}}a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right).$

160 3a - b + 2c + d.

161 4t - x - y - z.

162 2a - b - c - d.

163 -1,487,600.

164 -29,400,000.

165 48.

166 1.

167 160.

168 12.

169 900.

170 394.

171 665.

172 $a^2+b^2+c^2-2(bc+ca+ab)$. **173** $-2(x^3+y^3)$.
174 $(x+1)(x^2-x+1)^2$. **175** x^2z^2. **176** $-3(x^2-1)(x^2-4)$.
177 $\sin(c-a)\sin(c-b)\sin(a-b)$. **178** $(af-be+cd)^2$. **179** $n!$
180 $b_1b_2\dots b_n$. **181** $(x-x_1)(x-x_2)\dots(x-x_n)$.
182 $(n-1)!$ **183** $-2(n-2)!$ **184** 1.
185 $\frac{na^{n-1}}{2}[2a+(n-1)]h$. **186** $\frac{na^{n-1}}{2}[2a+(n-1)h]$.
187 $(-1)^{\frac{n(n+1)}{2}}[a_0-a_1+a_2-\dots+(-1)^n a_n]$.
188 $a_0x^n+a_1x^{n-1}+\dots+a_n$. **189** $\frac{nx^n}{x-1}-\frac{x^n-1}{(x-1)^2}$.
190 $(n+1)!\,x^n$. **191** $(x-a_1)(x-a_2)\dots(x-a_n)$.
192 $[x+(n-1)a](x-a)^{n-1}$. **193** $\frac{(x+a)^n+(x-a)^n}{2}$.
194 $(-1)^n(n+1)a_1a_2\dots a_n$.
195 $a_1a_2\dots a_n\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$.
196 $h(x+h)^n$. **197** $(-1)^{n-1}(n-1)x^{n-2}$.
198 $(-1)^n 2^{n-1}a_1a_2\dots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$.

199 $(-1)^{\frac{n(n-1)}{2}}\frac{n^{n-1}(n+1)}{2}$. **200** $\frac{(n+1)^{n+1}}{n^n}$.
201 $\prod_{k=1}^{n}(1-ax_{k,k})$. **202** $(-1)^{n-1}2^{n-2}(n+1)$.
203 $(-1)^n(a_0b_0+a_1b_1+\dots+a_nb_n)b_1b_2\dots b_{n-1}$.
204 $a(a+b)(a+2b)\dots[a+(n+1)b]$.

205 $x^n+(-1)^{n-1}y^n$.

206 0, if $n>2$.

207 0, if $n>2$.

208 For n = 2, we use the hints as follows:

$$\begin{vmatrix} 1+a_1+x_1 & a_1+x_2 \\ a_2+x_1 & 1+a_2+x_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a_1+x_2 \\ 0 & a_2+x_2 \end{vmatrix} + \begin{vmatrix} a_1+x_1 & 0 \\ a_2+x_1 & 1 \end{vmatrix}$$
$$+ \begin{vmatrix} a_1+x_1 & a_1+x_2 \\ a_2+x_1 & a_2+x_2 \end{vmatrix} = 1 + [(a_1+x_1)+(a_2+x_2)] + (a_2-a_1)(x_1-x_2).$$

In the general case, the n-th order determinant can be written as the sum of 2^n determinants; one of these has the value 1; n have the values $a_1 + x_1$, $a_2 + x_2, \cdots, a_n + x_n$; $n(n-1)/2$ have the respective values

$(a_i - a_k)(x_i - x_k)$, $i > k$.

The remaining terms have the value 0. Thus the result is

$$1 + \sum_{i=1}^{n} (a_i + x_i) + \sum_{i>k} (a_i - a_k)(x_k - x_i).$$

which can be written in the form

$$(1+a_1+a_2+\dots+a_n)(1+x_1+x_2+\dots+x_n) - \\ - n(a_1x_1+a_2x_2+\dots+a_nx_n).$$

209 0, if $p > 2$.

211 $\frac{n+1}{1-x} + \frac{x^{n+1}-1}{(1-x)^2}$.

212 Solution. It is clear that $\Delta_2 = x_1x_2\left(1 + \frac{a_1}{x_1} + \frac{a_2}{x_2}\right)$. Suppose

$$\Delta_{n-1} = x_1x_2 \dots x_{n-1}\left(1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} + \dots + \frac{a_{n-1}}{x_{n-1}}\right).$$

Then

$$\Delta_n = x_1 x_2 \dots x_n \left(1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} + \dots + \frac{a_{n-1}}{x_{n-1}}\right)$$
$$+ a_n x_1 x_2 \dots x_{n-1} = x_1 x_2 \dots x_n \left(1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} + \dots + \frac{a_n}{x_n}\right).$$

213 $a_0 x_1 x_2 \dots x_n + a_1 y_1 x_2 \dots x_n + a_2 y_1 y_2 x_3 \dots x_n + \dots +$
$+ a_n y_1 y_2 \dots y_n.$

214 $-a_1 a_2 \dots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right).$

215 $n!(a_0 x^n + a_1 x^{n-1} + \dots + a_n).$

216 $a_1 a_2 \dots a_{n-1} - a_1 a_2 \dots a_{n-2} + \dots + (-1)^n a_1 + (-1)^{n+1}.$

217 Solution. Expand by minors of the first column:

$\Delta_n = (\alpha + \beta)\Delta_{n-1} - \alpha\beta\Delta_{n-2}$. We check $\Delta_2 = \frac{\alpha^3 - \beta^3}{\alpha - \beta}$;

$\Delta_3 = \frac{\alpha^4 - \beta^4}{\alpha - \beta}$, and suppose $\Delta_{n-2} = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$;

$\Delta_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. Then

$$\Delta_n = (\alpha + \beta)\frac{\alpha^n - \beta^n}{\alpha - \beta} - \alpha\beta\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

Second Solution.

Write Δ_n as $d_n + \delta_n$, where

$$d_n = \det \begin{bmatrix} \alpha & \alpha\beta & 0 & \dots 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots 1 & \alpha+\beta \end{bmatrix},$$

$$\delta_n = \det \begin{bmatrix} \beta & 0 & 0 & \dots 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots 1 & \alpha+\beta \end{bmatrix}.$$

We treat these matrices as follows. Factor α from the first row of the first matrix; subtract the first row from the second. We obtain the result $d_n = \alpha d_{n-1}$. From the obvious relation $d_2 = \alpha^2$, we see that $d_n = \alpha^n$ (this is really established by induction).

Now we expand the second determinant by minors of the elements of the first row and obtain the relation

$$\delta_n = \beta\Delta_{n-1}.$$

We now notice that $\Delta_n = \alpha^n + \beta\Delta_{n-1}$. Next we check the relation $\Delta_2 = \frac{\alpha^3 - \beta^3}{\alpha - \beta}$, and use the inductive hypothesis $\Delta_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. It follows easily that $\Delta_n = \alpha^n + \beta \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$.

218 $n + 1$.

219 $\sin(n + 1)\theta/\sin\theta$.

220 $\cos n\theta$.

221 $x^n - C_1^{n-1} x^{n-2} + C_2^{n-2} x^{n-4} - \dots$ Cf. problem 53.

222 $x_1 y_n \prod_{i=1} (x_{i+1} y_i - x_i y_{i+1})$.

223 $a_1 a_2 \dots a_n \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)$.

224 $(-1)^{\frac{n(n-1)}{2}} a_1 a_2 \dots a_n \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)$.

225 $x(a_1 - x) \dots (a_n - x)\left(\frac{1}{x} + \frac{1}{a_1 - x} + \dots + \frac{1}{a_n - x}\right)$.

226 $(x_1-a_1)(x_2-a_2)\dots(x_n-a_n)\left(1+\frac{a_1}{x_1-a_1}+\dots+\right.$
$\left.+\frac{a_n}{x_n-a_n}\right).$

227 $\prod_{i=1}^{n}(x_i-a_ib_i)\left(1+\sum_{i=1}^{n}\frac{a_ib_i}{x_i-a_ib_i}\right).$

228 $(-1)^n m^n\left(1-\sum_{i=1}^{n}\frac{x_i}{m}\right).$

229 $x_1=x_2=\dots=x_{n-1}=0;\quad x_n=\sum_{i=1}^{n}\frac{a_i}{\alpha_i}.$

230 $(a^2-b^2)^n$. **231.** $a(a+b)\dots[a+(n-1)b]\left(\frac{1}{a}+\frac{1}{a+b}\right.$
$\left.+\dots+\frac{1}{a+(n-1)b}\right).$

232 $x^{n-1}\prod_{i=1}^{n}(x-2a_i)\left(x+\sum_{i=1}^{n}\frac{a_i^2}{x-2a_i}\right).$

233 $x^{n-1}\prod_{i=1}^{n}(x-2a_i)\left(x+\sum_{i=1}^{n}\frac{a_i^2}{x-2a_i}\right).$

234 $1-b_1+b_1b_2-b_1b_2b_3+\dots+(-1)^n b_1b_2\dots b_n.$

235 $(-1)^{n-1}(b_1a_2a_3\dots a_n+b_1b_2a_3\dots a_n+\dots+b_1b_2\dots b_{n-1}a_n).$

236 $(-1)^{n-1}x^{n-2}$. **237** $(-1)^n[(x-1)^n-x^n].$

238 $a_0x^n\prod_{i=1}^{n}(b_i-a_i)$. **240** 1. **241** 1. **242** 1.

243 $$\frac{C_{n+1}^{m+n}\cdot C_{n+1}^{m+n-1}\dots C_{n+1}^{m+n-k+1}}{C_{n+1}^{k+n}C_{n+1}^{k+n-1}\dots C_{n+1}^{n+1}}.$$

244 $(-1)^{\frac{m(m+1)}{2}}.$

245 $(x - 1)^n$

246 $(n - 1)!\,(n - 2)!\,\cdots 1!\,(x - 1)^n.$

247 $\alpha^n.$

248 The hints lead to

$$\Delta_n = (x - z)\,\Delta_{n-1} + z\,(x - y)^{n-1},$$
$$\Delta_n = (x - y)\,\Delta_{n-1} + y\,(x - z)^{n-1}.$$

Thus by elimination:

$$\Delta_n = \frac{z\,(x - y)^n - y\,(x - z)^n}{z - y}.$$

249 $(-1)^{\frac{n(n+1)}{2}}\,\dfrac{ab\,(b^{n-1} - a^{n-1})}{a - b}.$

250 $\dfrac{xf(y) - yf(x)}{x - y}$, where $f(x) = \prod_{k=1}^{n} (a_k - x).$

251 $\dfrac{f(a) - f(b)}{a - b}$, where $f(x) = \prod_{k=1}^{n} (c_k - x).$

252 $(\alpha - \beta)^{n-2}\,[\lambda\alpha + (n - 2)\,\lambda\beta - (n - 1)\,ab].$

253 $(-1)^{\frac{n(n-1)}{2}}\,\dfrac{n^{n-1}\,(n + 1)}{2}.$

254 $(-1)^{\frac{n(n-1)}{2}}\,(nh)^{n-1}\left[a + \dfrac{h\,(n - 1)}{2}\right].$

255 $(1 - x^n)^{n-1}.$

256 If we add the last three columns to the first we note that the determinant is a rational function of a that has the quantity $a + b + c + d$ as a factor.

We can show that the value of the determinant is divisible by $a + b - c - d$ as follows: To the first column add the second and subtract the

third and fourth. By a variation of this method, we see that the value of the determinant is also divisible by $a - b + c - d$, $a - b - c + d$. Thus the determinant has the form

$\lambda(a + b + c + d)(a + b - c - d)(a - b + c - d)$ $(a - b - c + d)$. Since the determinant is clearly a fourth-degree polynomial in a, λ must be a constant. The coefficient of a^4 is 1; therefore $\lambda = 1$.

257 $$\begin{aligned}&(a+b+c+d+e+f+g+h)(a+b+c+d-e-f-g-h)\\ &\times(a+b-c-d+e+f-g-h)(a+b-c-d-e-f+g+h)\\ &\times(a-b+c-d+e-f+g-h)(a-b+c-d-e+f-g+h)\\ &\times(a-b-c+d+e-f-g+h)(a-b-c+d-e+f+g-h).\end{aligned}$$

258 $$(x+a_1+a_2+\ldots+a_n)(x-a_1)(x-a_2)\ldots(x-a_n).$$

259 $$2^{\frac{n(n-1)}{2}}\prod_{1\leqslant i<k\leqslant n}\sin\frac{\varphi_i+\varphi_k}{2}\prod_{1\leqslant i<k\leqslant n}\sin\frac{\varphi_k-\varphi_i}{2}.$$

260 $$2^{\frac{n(n-1)}{2}}\prod_{n\geqslant i>k\geqslant 1}\cos\frac{\varphi_i+\varphi_k}{2}\prod_{n\geqslant i>k\geqslant 1}\sin\frac{\varphi_i-\varphi_k}{2}.$$

261 $1!\ 2!\cdots n!$.

262 $$\prod_{n+1\geqslant k>i\geqslant 1}(a_i-a_k).$$

263 $(-1)^n 1!\ 2!\cdots n!$

264 $$(-1)^{n-1}\prod_{i=1}^{n}a_i\prod_{n\geqslant i>k\geqslant 1}(a_i-a_k)\left(\sum_{i=1}^{n}\frac{w_i}{a_i f'(a_i)}\right),$$

where $f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$.

265 $\prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

266 $2^{\frac{n(n-1)}{2}} \prod_{n \geqslant i > k \geqslant 1} \cos\frac{\varphi_i + \varphi_k}{2} \prod_{n \geqslant i > k \geqslant 1} \sin\frac{\varphi_i - \varphi_k}{2}$.

267 $\prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

268 $2^{\frac{n(n-1)}{2}} a_{01}a_{02} \ldots a_{0n-1} \prod_{n \geqslant i > k \geqslant 1} \sin\frac{\varphi_i + \varphi_k}{2} \times$

$\times \prod_{n \geqslant i > k \geqslant 1} \sin\frac{\varphi_k - \varphi_i}{2}$.

269 $\frac{1}{1!\,2! \ldots (n-1)!} \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

271 $1!\,3!\,5! \ldots (2n-1)!$ **272** $\prod_{i=1}^{n} \frac{x_i}{x_i - 1} \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

273 $\prod_{n+1 \geqslant k > i \geqslant 1} (b_k a_i - a_k b_i)$. **274** $\prod_{1 \leqslant i < k \leqslant n} \sin(\alpha_i - \alpha_k)$.

275 $\prod_{1 \leqslant i < k \leqslant n+1} (a_i - a_k)(a_i a_k - 1)$.

276 $2^{(n-1)^2} \prod_{n-1 \geqslant i > k \geqslant 0} \sin\frac{\varphi_i + \varphi_k}{2} \prod_{n-1 \geqslant i > k \geqslant 0} \sin\frac{\varphi_k - \varphi_i}{2}$.

277 $2^{n(n+1)} \sin\alpha_0 \sin\alpha_1 \ldots$

$\ldots \sin\alpha_n \prod_{n \geqslant i > k \geqslant 0} \sin\frac{\alpha_i + \alpha_k}{2} \prod_{n \geqslant i > k \geqslant 0} \sin\frac{\alpha_i - \alpha_k}{2}$.

278 $[x_1 x_2 \ldots x_n - (x_1 - 1) \ldots (x_n - 1)] \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

279 $x_1 x_2 \ldots x_n \left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right) \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

280 $(x_1 + x_2 + \ldots + x_n) \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

281 $\sigma_{n-s} \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$, where σ_p denotes the sum of all possible products of $x_1, x_2, \cdots, x_n$, taken p at a time.

282 $$[2x_1x_2 \ldots x_n - (x_1 - 1)(x_2 - 1) \ldots (x_n - 1)] \times \times \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k).$$

283 $x^2(x^2 - 1)^4$.

284 $2x^3y(x-y)^6$.

285 $1!\,2!\,3!\ldots(n-1)!\,x^{\frac{n(n-1)}{2}}(y-x)^n$.

286 $$1!\,2!\,3!\ldots(k-1)!\,x^{\frac{k(k-1)}{2}}(y_1-x)^k(y_2-x)^k\ldots \ldots(y_{n-k}-x)^k \prod_{n-k \geqslant i > j \geqslant 1}(y_i - y_j).$$

287 $(y-x)^{k(n-k)}$.

288 b) 9; c) 5; e) 128; f) $(a_1a_2 - b_1b_2)(c_1c_2 - d_1d_2)$;
g) $(x_3 - x_2)^2(x_3 - x_1)^2(x_2 - x_1)^2$;
h) $(\lambda^2 - a^2)(\alpha - \beta)^{n-1}[\alpha + (n-1)\beta]$;
k) $(x_4 - x_3)[(x_3 - x_2)(x_4 - x_2) - 2(x_3 - x_1)(x_4 - x_1)]$;
m) $27(a+2)^3(a-1)^6[3(a+2)^2 - 4x^2][3(a-1)^2 - 4x^2]^2$.

Note: This problem is a special case of 537.

289

a) $\begin{vmatrix} -5 & -2 \\ -8 & 4 \end{vmatrix}$; b) $\begin{vmatrix} 2 & 8 & 17 \\ 11 & -6 & 5 \\ 3 & 8 & -3 \end{vmatrix}$;

c) $\begin{vmatrix} 7 & 5 & -3 & 3 \\ -1 & 5 & -3 & 3 \\ -4 & -4 & -5 & 4 \\ -4 & -4 & 0 & 3 \end{vmatrix}$.

290 a) 24; b) 18;

c) $(a + b + c + d)(a + b - c - d)(a - b + c - d)$
$(a - b - c + d)$.

291 a) 16; b) 280; c) $(a^2 + b^2 + c^2 + d^2)^2$.

292 $D \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)$.

293 a) $C_1^n C_2^n \cdots C_n^n \prod_{n \geqslant i > k \geqslant 0} (a_i - a_k)(b_k - b_i)$.

b) $\prod_{n \geqslant i > k \geqslant 1} (\alpha_i - \alpha_k)(\beta_i - \beta_k)$.

294 0, if $n \geq 2$.

295 $\prod_{i=1}^{n} (x - x_i) \prod_{n \geqslant i > k \geqslant 1} (x_i - x_k)^2$.

296 $-(a^2 + b^2 + c^2 + d^2 + l^2 + m^2 + n^2 + p^2)^4$.

297 $4 \sin^4 \varphi$.

298 $4 \sin^4 \varphi$.

299 Let the value of the determinant be Δ_n. By squaring the matrix involved, we see that $\det \Delta = n^{n/2}$. On the other hand,

$$\Delta = \prod_{n-1 \geqslant k > s \geqslant 0} (\varepsilon^k - \varepsilon^s).$$

Set $\varepsilon_1 = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$. Then $\varepsilon = \varepsilon_1^2$,

$$\Delta = \prod_{n-1 \geqslant k > s \geqslant 0} (\varepsilon^k - \varepsilon^s) = \prod \varepsilon_1^{k+s} \prod \left(\varepsilon_1^{k-s} - \varepsilon_1^{-k+s}\right) =$$

$$= \prod \varepsilon_1^{k+s} \cdot i^{\frac{n(n-1)}{2}} \prod 2\sin\frac{(k-s)\pi}{n}.$$ In fact $\sin\frac{(k-s)\pi}{n} > 0$, for all k, s. Thus

$$n^{\frac{n}{2}} = |\Delta| = \left|\prod 2\sin\frac{(k-s)\pi}{n}\right| = \prod 2\sin\frac{(k-s)\pi}{n}.$$

and

$$\Delta = n^{\frac{n}{2}} i^{\frac{n(n-1)}{2}} \prod_{n-1 \geqslant k > s \geqslant 0} \varepsilon_1^{k+s} = n^{\frac{n}{2}} i^{\frac{n(n-1)}{2}} \varepsilon_1^{\frac{n(n-1)^2}{2}} =$$

$$= n^{\frac{n}{2}} i^{\frac{n(n-1)}{2} + (n-1)^2} = n^{\frac{n}{2}} i^{-\frac{(n-1)(n-2)}{2}}.$$

300 $\prod_{k=0}^{n-1} (a_0 + a_1\varepsilon_k + a_2\varepsilon_k^2 + \ldots + a_{n-1}\varepsilon_k^{n-1})$, where $\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

301 $x^4 - y^4 + z^4 - u^4 + 4xy^2z + 4xzu^2 - 4x^2yu - 4yz^2u - 2x^2z^2 + 2y^2u^2$.

303 2^{n-1}, if n is odd; 0, if n is even.

304 $(-1)^n \frac{[(n+1)a^n - 1]^n - n^n a^{n(n+1)}}{(1-a^n)^2}$.

305 $(-1)^{n-1}(n-1)\prod_{k=0}^{n-1}(a_1+a_2\varepsilon_k+\dots+a_n\varepsilon_k^{n-1})$, where

$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

306 $\varphi_0(t)\varphi_1(t)\dots\varphi_{n-1}(t)$, where $\varphi_k(t)=\frac{(t+\varepsilon_k)^n-1}{t}$;

$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

By problem 102, the result can be written

$$\prod_{k=1}^{n-1}[t^n-(\varepsilon_k-1)^n].$$

307 $(-2)^{n-1}(n-2p)$, if $(n,p)=1$; 0, if $(n,p)\neq 1$.

308 $2^{n-2}\left(\cos^n\frac{\pi}{n}-1\right)$.

309 $2^{n-2}\sin^{n-2}\frac{n\theta}{2}\left[\sin^n\frac{(n+2)\theta}{2}-\sin^n\frac{n\theta}{2}\right]$.

310 $(-1)^n 2^{n-2}\sin^{n-2}\frac{nh}{2}\left[\cos^n\left(a+\frac{nh}{2}\right)-\cos^n\left(a+\frac{(n-2)h}{2}\right)\right]$.

311 $(-1)^{n-1}\frac{(n+1)(2n+1)}{12}n^{n-2}[(n+2)^n-n^n]$.

313 $\prod_{k=0}^{n-1}(a_1+a_2\varepsilon_k+a_3\varepsilon_k^2+\dots+a_n\varepsilon_k^{n-1})$,

where $\varepsilon_k=\cos\frac{(2k+1)\pi}{n}+i\sin\frac{(2k+1)\pi}{n}$.

315 $\prod_{i=1}^{n}(a_1+a_2\rho_i+a_3\rho_i^2+\dots+a_n\rho_i^{n-1})$,

where $\rho_1, \rho_2, \dots, \rho_n$ are n-th roots of μ.

318 As for 223: If we add 1 to each element of the matrix

$$\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

we obtain the matrix of 223.

Thus $\Delta = a_1 a_2 \dots a_n + \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik};$

$$\sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik} = a_1 a_2 \dots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

As for 250: the determinant Δ satisfies:

$$\Delta = (a_1 - x)(a_2 - x) \dots (a_n - x) + x \sum A_{ik};$$
$$\Delta = (a_1 - y)(a_2 - y) \dots (a_n - y) + y \sum A_{ik},$$

where $\sum A_{ik}$ is the sum of the algebraic cofactors of all the elements of Δ. Using these equations the value of Δ is easily determined.

323 $$\prod_{1 \leqslant i < k \leqslant n} (a_i - a_k) \prod_{1 \leqslant i < k \leqslant n} (b_i - b_k) \cdot \frac{1}{\prod_{i=1}^{n} f(a_i)},$$

where $f(x) = (x + b_1) \cdots (x + b_n)$.

325 $$\frac{[c + \sqrt{c^2 - 4ab}]^{n+1} - [c - \sqrt{c^2 - 4ab}]^{n+1}}{2^{n+1} \sqrt{c^2 - 4ab}}.$$

326 $$\frac{[p + \sqrt{p^2 - 4q}]^n + [p - \sqrt{p^2 - 4q}]^n}{2^n}.$$

327 $x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$, where a_k is the sum of all the k-th order minors of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

that are contained in the n - k rows with indices $\alpha_1, \alpha_2, \dots, \alpha_{n-k}$, and columns with the same indices.

328 $(n+1)^{n-1}$.

329 $(x-n)^{n+1}$.

330 $(x^2 - 1^2)(x^2 - 3^2) \dots [x^2 - (2m-1)^2]$, if n = 2m;

$x(x^2 - 2^2)(x^2 - 4^2) \dots (x^2 - 4m^2)$, if n = 2m + 1.

331 $(x + na - n)[x + (n-2)a - n + 1] \times$
$\times [x + (n-4)a - n + 2] \dots (x - na)$.

332 $(-1)^{\frac{n(n-1)}{2}} [(n-1)!]^n$.

333 $\dfrac{[1!2!\dots(n-1)!]^3}{n!(n+1)!\dots(2n-1)!}$.

334 $\dfrac{\Delta(a_1, a_2, \dots, a_n)\,\Delta(b_1, b_2, \dots, b_n)}{\Delta(1, 2, \dots, n)}$, where Δ is the Vandermonde determinant.

CHAPTER III - SOLUTIONS

SYSTEMS OF LINEAR EQUATIONS

335 $x_1 = 3; \quad x_2 = x_3 = 1.$ **336** $x_1 = 1; \quad x_2 = 2; \quad x_3 = -2.$

337 $x_1 = 2; \; x_2 = -2; \; x_3 = 3.$ **338** $x_1 = 3; \; x_2 = 4; \; x_3 = 5.$

339 $x_1 = x_2 = -1; \quad x_3 = 0; \quad x_4 = 1.$

340 $x_1 = 1; \quad x_2 = 2; \quad x_3 = -1; \; x_4 = -2.$

341 $x_1 = -2; \quad x_2 = 2; \quad x_3 = -3; \quad x_4 = 3.$

342 $x_1 = 1; \quad x_2 = 2; \quad x_3 = 1; \quad x_4 = -1.$

343 $x_1 = 2 \quad x_2 = x_3 = x_4 = 0.$ **344** $x_1 = x_2 = x_3 = x_4 = 0.$

345 $x_1 = 1; \quad x_2 = -1; \quad x_3 = 0; \quad x_4 = 2.$

346 $x_1 = x_2 = x_3 = x_4 = x_5 = 0.$ **347** $x_1 = x_2 = x_2 = x_4 = 0.$

348 $x_1 = 1; \quad x_2 = -1; \quad x_3 = 1; \quad x_4 = -1; \quad x_5 = 1.$

349 $x_1 = x_2 = x_3 = x_4 = x_5 = 0.$

350 $x_1 = 1; \quad x_2 = -1; \quad x_3 = 1; \quad x_4 = -1; \quad x_5 = 1.$

351 $x_1 = 0; \quad x_2 = 2; \quad x_3 = -2; \quad x_4 = 0; \quad x_5 = 3.$

352 $x_1 = 2; \quad x_2 = 0; \quad x_3 = -2; \quad x_4 = -2; \quad x_5 = 1.$

353 Since the system has a non-trivial solution, the determinant of the matrix of coefficients must be zero.

354 The determinant of the matrix of coefficients is $-(a^2 + b^2 + c^2 + d^2)^2$.

355 $$x_i = \frac{\alpha \sum_{k=1}^{n} a_k - a_i[(n-1)\alpha + \beta]}{(\alpha - \beta)[(n-1)\alpha + \beta]}.$$

356 $x_i = -\dfrac{f(\beta_i)}{\varphi'(\beta_i)}$, where $\begin{array}{l} f(x) = (x-b_1)(x-b_2)\dots(x-b_n). \\ \varphi(x) = (x-\beta_1)(x-\beta_2)\dots(x-\beta_n). \end{array}$

357 $x_i = \dfrac{f(t)}{(t-\alpha_i)f'(\alpha_i)}$, where $f(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$.

358 $x_s = \sum\limits_{i=1}^{n} \dfrac{(-1)^{n+s} u_i}{f'(\alpha_i)} \varphi_{s,i}$, where $f(x) = (x-\alpha_1)\dots(x-\alpha_n)$;
$\varphi_{s,i} = \sum \alpha_{t_1}\alpha_{t_2}\dots\alpha_{t_{n-s}}$,

where the sum is taken over all subsets $t_1, t_2, \dots, t_{n-s}$ of $1, 2, \cdots, i-1, i+1, \cdots, n$.

359 $x_s = \sum\limits_{i=1}^{n} \dfrac{(-1)^{n+i} u_i}{f'(\alpha_s)} \varphi_{i,s}$, where $f(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$;
$\varphi_{i,s} = \sum \alpha_{t_1}\alpha_{t_2}\dots\alpha_{t_{n-i}}$;
where the sum is taken over

all subsets $t_1, t_2, \dots, t_{n-i}$ of $1, 2, \cdots, s-1, s+1, \cdots, n$.

360 $x_i = \dfrac{(-1)^n a_{n-i}}{n!}$, where $x^n + a_1x^{n-1} + \dots + a_n = (x-1)(x-2)\dots(x-n)$.

361 $C_k^m \cdot C_k^n$.

365 a) Either unchanged or increased by unity.

b) Either unchanged or increased by one or two units.

366 2.

367 3.

368 2.

369 2.

370 3.

371 3.

372 4.

373 3.

374 2.

375 3.

376 5.

377 6.

378 5.

379 3.

380 4.

383 The forms are independent.

384 $2y_1 - y_2 - y_3 = 0.$

385 $y_1 + 3y_2 - y_3 = 0; \quad 2y_1 - y_2 - y_4 = 0.$

386 The forms are independent.

387 $y_1 + y_2 - y_3 - y_4 = 0.$

388 $y_1 - y_2 + y_3 = 0; \quad 5y_1 - 4y_2 + y_4 = 0.$

389, 390 The forms are independent.

391 $y_1 + y_2 - y_3 - y_4 = 0.$

392 $2y_1 - y_2 - y_3 = 0.$

393 $3y_1 - y_2 - y_3 = 0; \quad y_1 - y_2 - y_4 = 0.$

394 The forms are independent.

395 $y_1 - y_2 - y_3 - y_4 = 0.$

396 $3y_1 - 2y_2 - y_3 + y_4 = 0; \quad y_1 - y_2 + 2y_3 - y_5 = 0.$

397 $\lambda = 10; \quad 3y_1 + 2y_2 - 5y_3 - y_4 = 0.$

398 $x_3 = 2x_2 - x_1; \quad x_4 = 1.$

399 $\lambda = 5.$

400 No solution.

401 $x_1 = 1;\ x_2 = 2;\ x_3 = -2.$

402 $x_1 = 1; \quad x_2 = 2; \quad x_3 = 1.$

403 $x_1 = -\frac{11x_3}{7};\ x_2 = -\frac{x_3}{7}.$

404 No solution.

405 $x_1 = 0; \quad x_2 = 2; \quad x_3 = \frac{5}{3};\ x_4 = -\frac{4}{3}.$

406 $x_1 = -8;\quad x_2 = 3 + x_4;\quad x_3 = 6 + 2x_4.$

407 $x_1 = 2;\quad x_2 = x_3 = x_4 = 1.$

408 $x_1 = x_2 = x_3 = x_4 = 0.$

409 $x_1 = \frac{3x_3 - 13x_4}{17};\quad x_2 = \frac{19x_3 - 20x_4}{17}.$

410 $x_1 = \frac{7}{6}x_5 - x_3;\quad x_2 = \frac{5}{6}x_5 + x_3;\quad x_4 = \frac{x_5}{3}.$

411 $x_1 = -16 + x_3 + x_4 + 5x_5;\quad x_2 = 23 - 2x_3 - 2x_4 - 6x_5.$

412 $x_1 = \frac{-4x_4 + 7x_5}{8};\quad x_2 = \frac{-4x_4 + 5x_5}{8};\quad x_3 = \frac{4x_4 - 5x_5}{8}.$

413 $x_1 = x_2 = x_3 = 0;\quad x_4 = x_5.$

414 $x_1 = \frac{1 + x_5}{3};\quad x_2 = \frac{1 + 3x_3 + 3x_4 - 5x_5}{3}.$

415 $x_1 = \frac{2 + x_5}{3};\quad x_2 = \frac{1 + 3x_3 - 3x_4 + 5x_5}{6}.$

416, 417 No solution.

418 $x_1 = -\frac{x_5}{2};\quad x_2 = -1 - \frac{x_5}{2};\quad x_3 = 0;\quad x_4 = -1 - \frac{x_5}{2}.$

419 $x_1 = \frac{1 + 5x_4}{6};\quad x_2 = \frac{1 - 7x_4}{6};\quad x_3 = \frac{1 + 5x_4}{6}.$

420 No solution.

421 $x = \frac{b^2 + c^2 - a^2}{2bc};\quad y = \frac{a^2 + c^2 - b^2}{2ac};\quad z = \frac{a^2 + b^2 - c^2}{2ab}.$

422 If $(\lambda - 1)(\lambda + 2) \neq 0,\ x = -\frac{\lambda + 1}{\lambda + 2};\quad y = \frac{1}{\lambda + 2};$

$$z = \frac{(\lambda + 1)^2}{\lambda + 2}.$$

If $\lambda = 1$, the solution of the system depends on two independent parameters.

If $\lambda = -2$, there is no solution.

423 If $(\lambda-1)(\lambda+3)\neq 0$, then $x=-\frac{\lambda^2+2\lambda+2}{\lambda+3}$;
$y=-\frac{\lambda^2+\lambda-1}{\lambda+3}$; $z=\frac{2\lambda+1}{\lambda+3}$; $t=\frac{\lambda^3+3\lambda^2+2\lambda+1}{\lambda+3}$.

If $\lambda = 1$, the solution depends on three independent parameters.

If $\lambda = -3$, there is no solution.

424 If a, b, c are all distinct,

$$x = abc; \quad y = -(ab + ac + bc); \quad z = a + b + c.$$

If exactly two of a, b, c are equal the solution depends on one parameter.

If a = b = c, the solution set is expressed linearly in terms of two independent parameters.

425 If a, b, c are all distinct, then

$$x=\frac{(b-d)(c-d)}{(b-a)(c-a)}; \quad y=\frac{(a-d)(c-d)}{(a-b)(c-b)}; \quad z=\frac{(a-d)(b-d)}{(a-c)(b-c)}.$$

If a = b; a ≠ c; d = a or d = c, the solution set is a one-parameter set.

If b = c; a ≠ b; d = a or d = b, the solution set is a one-parameter set.

If a = c; a ≠ b; d = a or d = b, the solution set is one-parameter set.

If a = b = c = d, the solution set is a two-parameter set.

In all remaining cases, the system has no solution.

426 If $b(a-1) \neq 0$, then

$$x = \frac{2b-1}{b(a-1)};\ y = \frac{1}{b};\ z = \frac{2ab-4b+1}{b(a-1)}.$$

If $a = 1$; $b = 1/2$, the solution set is a one-parameter set.

In any other case, there is no solution.

427 If $b(a-1)(a+2) \neq 0$, then $x = z = \frac{a-b}{(a-1)(a+2)}$;

$$y = \frac{ab+b-2}{b(a-1)(a+2)}.$$

If $a = -2$; $b = -2$, the solution depends on one parameter.

If $a = 1$; $b = 1$, the solution depends on two parameters.

No solution in the remaining cases.

428 If $(\alpha-1)(\alpha+2) \neq 0$, then $x = \frac{m\alpha+m-n-p}{(\alpha+2)(\alpha-1)}$;

$$y = \frac{n\alpha+n-m-p}{(\alpha+2)(\alpha-1)};\quad z = \frac{p\alpha+p-m-n}{(\alpha+2)(\alpha-1)}.$$

If $\alpha = -2$ and $m + n + p = 0$, the solution depends on one parameter.

If $\alpha = 1$ and $m = n = p$, the solution depends on two parameters.

No solution in the remaining cases.

429 If $a(a-b) \neq 0$, then

$$x = \frac{a^2(b-1)}{b-a};\ y = \frac{b(a^2-1)}{a(a-b)};\ z = \frac{a-1}{a(b-a)}.$$

If $a = b = 1$, the solution depends on two parameters.

No solution in the remaining cases.

430 $\Delta = \lambda^2(\lambda - 1)$. For $\lambda = 0$; $\lambda = 1$ the system is inconsistent (has no solution).

431 $\Delta = -2\lambda$. If $\lambda \neq 0$, then $x = 1 - \lambda$, $y = \lambda$, $z = 0$. If $\lambda = 0$, then $x = 1$, $z = 0$, and y is arbitrary.

432 $\Delta = (k - 1)^2(k + 1)$. If $k = 1$, the solution set is a one-parameter set. If $k = -1$, the system is inconsistent.

433 $\Delta = a(b - 1)(b + 1)$.
If $a = 0$, $b = 5$; then $y = -1/3$, $z = 4/3$, with x arbitrary.
If $a = 0$, $b \neq 1$, $b \neq 5$, the system is inconsistent.
If $b = 1$, then $z = 0$, $y = 1 - ax$, x arbitrary.
If $b = -1$, the system is inconsistent.

434 a) $\Delta = -m(m + 2)$. For $m = 0$, $m = -2$ the system is inconsistent.
b) $\Delta = m(m^2 - 1)$. If $m = 0$ or $m = 1$ the system is inconsistent. If $m = -1$ the solution set is a one parameter set.
c) $\Delta = \lambda(\lambda - 1)(\lambda + 1)$. If $\lambda = 1$ or $\lambda = -1$ the system is inconsistent. If $\lambda = 0$ the solution set is a one parameter set.

435 a) $\Delta = 3(c + 1)(c - 1)^2$. If $c = -1$, the system is inconsistent. If $c = 1$, the solution set is a two-parameter set.
b) $\Delta = (\lambda - 1)(\lambda - 2)(\lambda - 3)$. If $\lambda = 2$ or $\lambda = 3$, the system is inconsistent. If $\lambda = 1$, the solution set is a one parameter set.

c) $\Delta = d(d-1)(d+2)$. If $d = 1$ or $d = -2$, the system is inconsistent. If $d = 0$ the solution set is a one parameter set.

d) $\Delta = (a-1)^2(a+1)$. If $a = -1$, the system is inconsistent. If $a = 1$, the solution set is a two parameter set.

436

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

437 If and only if

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

438

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0.$$

439 Only if

$$\det \begin{bmatrix} x_0^2+y_0^2 & x_0 & y_0 & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{bmatrix} = 0.$$

440 $(x-1)^2+(y-1)^2=1.$

441 $y^2-y=0.$

442 $y=x^3-1.$

443

$$\det \begin{bmatrix} y & x^n & x^{n-1} & \dots & x^2 & x & 1 \\ y_0 & x_0^n & x_0^{n-1} & \dots & x_0^2 & x_0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_n & x_n^n & x_n^{n-1} & \dots & x_n^2 & x_n & 1 \end{bmatrix} = 0.$$

444 If and only if

$$\det \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0.$$

445 $x^2 + y^2 + z^2 - x - y - z = 0.$

446 If and only if the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ & \cdots & \\ x_n & y_n & 1 \end{pmatrix}$$

is less than three.

447 If and only if the rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ & \cdots & \\ a_n & b_n & c_n \end{pmatrix}$$

is less than three.

448 In the same plane, if and only if the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n & y_n & z_n & 1 \end{pmatrix}$$

is less than four; on a single line if and only if the rank of the same matrix is less than three.

449 All the planes pass through a single point, if the rank of the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ \cdot & \cdot & \cdot & \cdot \\ A_n & B_n & C_n & D_n \end{pmatrix}$$

is less than four; through a single line if the rank of the same matrix is less than three.

450

$$\det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,\,n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,\,n-1} & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{n,\,n-1} & a_{nn} \end{bmatrix} = 0.$$

453 No.

454 For example,

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 5 & -6 & 0 & 0 & 1 \end{pmatrix}.$$

455 Yes.

456 Solution. Set

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{r1} & \alpha_{r2} & \dots & \alpha_{rn} \end{pmatrix}; \qquad B = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1r} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{r1} & \lambda_{r2} & \dots & \lambda_{rr} \end{pmatrix};$$

$$BA = \begin{pmatrix} \sum_{s=1}^{r} \lambda_{1s}\alpha_{s1} & \dots & \sum_{s=1}^{r} \lambda_{1s}\alpha_{sn} \\ \cdot & \cdot & \cdot \\ \sum_{s=1}^{r} \lambda_{rs}\alpha_{s1} & \dots & \sum_{s=1}^{r} \lambda_{rs}\alpha_{sn} \end{pmatrix}.$$

It is immediately obvious that the rows of the matrix BA form solutions of the system. Moreover since det B $\neq 0$, we have $A = B^{-1}(BA)$. This says that the solutions making up the rows of the matrix A are linear combinations of the solutions that make up the rows of the matrix BA.

457 Solution. Let

$$A=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n}\\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n}\\ \cdot & \cdot & \cdot & \cdot\\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rn}\end{pmatrix}; \qquad C=\begin{pmatrix}\gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n}\\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n}\\ \cdot & \cdot & \cdot & \cdot\\ \gamma_{r1} & \gamma_{r2} & \cdots & \gamma_{rn}\end{pmatrix}.$$

Since C is a set of fundamental solutions, we have $\alpha_{11}=\lambda_{11}\gamma_{11}+\lambda_{12}\gamma_{21}+\cdots+\lambda_{1r}\gamma_{r1}$, etc. Thus A = BC, where

$$B=\begin{pmatrix}\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1r}\\ \cdot & \cdot & \cdot & \cdot\\ \lambda_{r1} & \lambda_{r2} & \cdots & \lambda_{rr}\end{pmatrix}.$$

On the other hand A is also a set of fundamental solutions of system, and therefore det B $\neq 0$.

459 For example

$x_1=c_1+c_2+5c_3;\ x_2=-2c_1-2c_2-6c_3;\ x_3=c_1;\ x_4=c_2;\ x_5=c_3.$

See the solution to problem 454.

460 $x_1=11c;\quad x_2=c;\quad x_3=-7c.$

461 № 408. $x_1 = x_2 = x_3 = x_4 = 0$.

№ 409. $x_1 = 3c_1 + 13c_2$; $x_2 = 19c_1 + 20c_2$; $x_3 = 17c_1$; $x_4 = -17c_2$.
№ 410. $x_1 = c_1 + 7c_2$; $x_2 = -c_1 + 5c_2$; $x_3 = -c_1$; $x_4 = 2c_2$; $x_5 = 6c_2$;
№ 412. $x_1 = c_1 + 7c_2$; $x_2 = c_1 + 5c_2$; $x_3 = -c_1 - 5c_2$; $x_4 = -2c_1$; $x_5 = 8c_2$.
№ 413. $x_1 = 0$; $x_2 = 0$; $x_3 = 0$; $x_4 = c$; $x_5 = c$.

462 $x_1 = -16 + c_1 + c_2 + 5c_3$;
$x_2 = 23 - 2c_1 - 2c_2 - 6c_3$;
$x_3 = c_1$; $x_4 = c_2$; $x_5 = c_3$.

463 № 406. $x_1 = -8$; $x_2 = 3 + c$; $x_3 = 6 + 2c$; $x_4 = c$.
№ 414. $x_1 = c_3$; $x_2 = 2 + c_1 + c_2 - 5c_3$; $x_3 = c_1$; $x_4 = c_2$; $x_5 = -1 + 3c_3$.
№ 415. $x_1 = 1 + 2c_3$; $x_2 = 1 + c_1 - c_2 + 5c_3$; $x_3 = 2c_1$; $x_4 = 2c_2$; $x_5 = 1 + 6c_3$.

CHAPTER IV - SOLUTIONS

MATRICES

464 a) $\begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix}$; b) $\begin{pmatrix} -9 & 13 \\ 15 & 4 \end{pmatrix}$; c) $\begin{pmatrix} 6 & 2 & -1 \\ 6 & 1 & 1 \\ 8 & -1 & 4 \end{pmatrix}$;

d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; e) $\begin{pmatrix} 1 & 9 & 15 \\ -5 & 5 & 9 \\ 12 & 26 & 32 \end{pmatrix}$;

f) $\begin{pmatrix} a+b+c & a^2+b^2+c^2 & b^2+2ac \\ a+b+c & b^2+2ac & a^2+b^2+c^2 \\ 3. & a+b+c & a+b+c \end{pmatrix}$.

465 a) $\begin{pmatrix} 7 & 4 & 4 \\ 9 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}$; b) $\begin{pmatrix} 15 & 20 \\ 20 & 35 \end{pmatrix}$; c) $\begin{pmatrix} 3 & -2 \\ 4 & 8 \end{pmatrix}$;

d) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$; e) $\begin{pmatrix} \cos n\varphi & -\sin n\varphi \\ \sin n\varphi & \cos n\varphi \end{pmatrix}$.

466 $$\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix} = \sqrt{1+\frac{\alpha^2}{n^2}} \cdot \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$, where $\operatorname{tg}\varphi = \alpha/n$.

Thus
$$\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n = \left(1+\frac{\alpha^2}{n^2}\right)^{\frac{n}{2}} \cdot \begin{pmatrix} \cos n\varphi & \sin n\varphi \\ -\sin n\varphi & \cos n\varphi \end{pmatrix}.$$
The limit of the first factor is 1. Also

$\lim\limits_{n\to\infty} n\varphi = \alpha \lim\limits_{\varphi\to 0}\frac{\varphi}{\operatorname{tg}\varphi} = \alpha$. Thus

$$\lim\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

467 a) $(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$;

b) $(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2$.

c) Proof by induction.

468 a) $\begin{pmatrix} -10 & -4 & -7 \\ 6 & 14 & 4 \\ -7 & 5 & -4 \end{pmatrix}$; b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

469

a) $\begin{pmatrix} x & 2y \\ -y & x-2y \end{pmatrix} = (x-y)E + yA$;

b) $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = (x-y)E + yA$;

c) $\begin{pmatrix} x & y & 0 \\ u & v & 0 \\ 3t-3x-u & t-3y-v & t \end{pmatrix}$.

470 a) $\begin{pmatrix} 5 & 1 & 3 \\ 8 & 0 & 3 \\ -2 & 1 & -2 \end{pmatrix}$; b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

471 Use direct computation.

472 First note that there must exist polynomials of degree n^2 that are annihilated by the matrix A. For the equation $F(A) = a_0E + a_1A + \dots + a_mA^m = 0$ amounts to a system of n^2 linear homogeneous

equations in m + 1 independent variables $a_0, a_1, \cdots, a_m$. This system has a non-trivial solution for $m \geq n^2$. Now let F(x) be an arbitrary polynomial annihilated by A: F(A) = 0. Let f(x) be some polynomial of lowest possible degree that has the same property. Using the Euclidean algorithm we find that F(x) = f(x) q(x) + r(x), where the degree of r(x) is less than the degree of f(x). But since r(A) = F(A) - f(A) q(A) = 0, the minimal property of f(x) shows that r cannot be a non-trivial polynomial: $r(x) \equiv 0$. Finally F(x) = f(x) q(x).

f is the minimal polynomial of A.

473 Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad B=\begin{pmatrix} b_{11} & \dots & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{nn} \end{pmatrix}.$$

Then the sum of the diagonal elements of the matrix AB is $\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}$. One finds that the sum of the diagonal elements of the matrix BA is the same. Therefore, the sum of the diagonal elements of the matrix AB - BA must be zero, and the equation AB - BA = **E** is impossible.

Remark. This result is not valid for matrices with elements in a field of nonzero characteristic p. In fact for a field of characteristic p, the

equation AB - BA = E is satisfied by the following two p-th order matrices:

$$A=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 & 0 & \dots & & 0 \\ 1 & 0 & 0 & \dots & & 0 \\ 0 & 2 & 0 & \dots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & p-1 & 0 \end{pmatrix}.$$

474 $(E-A)(E+A+A^2+\dots+A^{k-1})=E-A^k=E.$

475 $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad bc=-a^2.$

476 If $A^3 = 0$, then $A^2 = 0$. For if $A^3 = 0$, then det A = 0. Therefore, see problem 471, $A^2 = (a + d)A$; $0 = A^3 = (a + d) A^2 = (a + d)^2 A$. Therefore $a + d = 0$, $A^2 = 0$.

477 $\pm E$; $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2=1-bc.$

478 If A = 0, then X can be arbitrary. If det A $\neq$ 0, then X = 0. Finally, if det A = 0, but A $\neq$ 0, then the rows of A are proportional. Let $\alpha : \beta$ be the ratio of corresponding elements of the first and second rows of A. Then $X=\begin{pmatrix} -\beta x & \alpha x \\ -\beta y & \alpha y \end{pmatrix}$ where x, y are arbitrary.

479 Set $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1) If A $\neq$ 0 but a + d = 0, ad - bc = 0, there is no solution.

2) If $a + d \neq 0$, $(a - d)^2 + 4bc = 0$; $(a - d)$, b, c are not identically zero, then there are two solutions:

$$X = \pm \frac{1}{2\sqrt{2(a+d)}} \begin{pmatrix} 3a+d & 2b \\ 2c & a+3d \end{pmatrix};$$

3) If $a + d \neq 0$, $ad - bc = 0$, there are two solutions:

$$X = \pm \frac{1}{\sqrt{a+d}} \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

4) If $ad - bc \neq 0$, $(a - d)^2 + 4bc \neq 0$, there are four solutions:

$$X = \frac{1}{\lambda} \begin{pmatrix} \frac{\lambda^2 + a - d}{2} & b \\ c & \frac{\lambda^2 - a + d}{2} \end{pmatrix},$$

where

$$\lambda = \pm \sqrt{a + d \pm 2\sqrt{ad - bc}};$$

5) If $a - d = b = c = 0$, there are infinitely many solutions:

$$X = \pm \sqrt{a}E \quad \text{or} \quad X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$$, where x, y, z

satisfy the relation $x^2 + yz = a$.

480 a) $\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$; b) $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$;

c) $\begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$; d) $\begin{pmatrix} 1 & -3 & 11 & -38 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$;

e) $\begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$; f) $\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$;

g) $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 31 & -19 & 3 & -4 \\ -23 & 14 & -2 & 3 \end{pmatrix}$;

h) $$\frac{1}{n-1}\begin{Vmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & 2-n & 1 & \dots & 1 \\ 1 & 1 & 2-n & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \dots & 2-n \end{Vmatrix}$$

i) $$\frac{1}{n}\begin{Vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon^{-1} & \varepsilon^{-2} & \dots & \varepsilon^{-n+1} \\ 1 & \varepsilon^{-2} & \varepsilon^{-4} & \dots & \varepsilon^{-2n+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{-n+1} & \varepsilon^{-2n+2} & \dots & \varepsilon^{-(n-1)^2} \end{Vmatrix};$$

j) $$\frac{1}{n+1}\begin{Vmatrix} 1\cdot n & 1\cdot(n-1) & 1\cdot(n-2) & \dots & 1\cdot 1 \\ 1\cdot(n-1) & 2\cdot(n-1) & 2\cdot(n-2) & \dots & 2\cdot 1 \\ 1\cdot(n-2) & 2\cdot(n-2) & 3\cdot(n-2) & \dots & 3\cdot 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1\cdot 1 & 2\cdot 1 & 3\cdot 1 & \dots & n\cdot 1 \end{Vmatrix};$$

k) $$\frac{1}{2n^3}\begin{pmatrix} 2-n^2 & 2+n^2 & 2 & \dots & 2 \\ 2 & 2-n^2 & 2+n^2 & \dots & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2+n^2 & 2 & 2 & \dots & 2-n^2 \end{pmatrix};$$

l) $$\frac{1}{d}\begin{Vmatrix} b_1c_1+d & b_2c_1 & \dots & b_nc_1 & -c_1 \\ b_1c_2 & b_2c_2+d & \dots & b_nc_2 & -c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_1c_n & b_2c_n & \dots & b_nc_n+d & -c_n \\ -b_1 & -b_2 & \dots & -b_n & 1 \end{Vmatrix},$$

where $d = a - b_1c_1 - b_2c_2 - \dots - b_nc_n$;

$$\text{m)}\ \frac{1}{f}\begin{pmatrix} f-f_0x^n & xf-f_1x^n & \dots & x^{n-1}f-f_{n-1}x^n & x^n \\ -f_0x^{n-1} & f-f_1x^{n-1} & \dots & x^{n-2}f-f_{n-1}x^{n-1} & x^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -f_0x & -f_1x & \dots & f-f_{n-1}x & x \\ -f_0 & -f_1 & \dots & -f_{n-1} & 1 \end{pmatrix},$$

where $f_0=a_0$, $f_1=a_0x+a_1$, $\dots$, $f_{n-1}=a_0x^{n-1}+\dots+a_{n-1}$, $f=a_0x^n+a_1x^{n-1}+\dots+a_n$;

$$\text{n)}\ \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}-\frac{1}{\mu}\begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_n \\ \lambda_2\lambda_1 & \lambda_2^2 & \dots & \lambda_2\lambda_n \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_n\lambda_1 & \lambda_n\lambda_2 & \dots & \lambda_n^2 \end{pmatrix},$$

$$\mu=1+\lambda_1+\lambda_2+\dots+\lambda_n;$$

$$\text{o)}\ \begin{pmatrix} B^{-1}+\lambda B^{-1}UVB^{-1} & -\lambda B^{-1}U \\ -\lambda VB^{-1} & \lambda \end{pmatrix},\quad \lambda=\frac{1}{a-VB^{-1}U}.$$

481 a) $\begin{pmatrix} 2 & -23 \\ 0 & 8 \end{pmatrix}$; b) $\begin{pmatrix} -3 & 2 & 0 \\ -4 & 5 & -2 \\ -5 & 3 & 0 \end{pmatrix}$;

$$\text{c)}\ \begin{pmatrix} 1 & -1 & -1 & 0 & \dots & & & 0 & 0 \\ 1 & 1 & -1 & -1 & \dots & & & 0 & 0 \\ 0 & 1 & 1 & -1 & \dots & & & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \end{pmatrix};$$

d) $\begin{pmatrix} 24 & 13 \\ -34 & -18 \end{pmatrix}$; e) $X = E - \frac{n-1}{n^2}\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \dots & 1 \end{pmatrix}$;

f) $X = \begin{pmatrix} 1+a & b \\ -2a & 1-2b \end{pmatrix}$; g) no solution.

482 Multiply both sides of the given equality on the left by A^{-1}.

483 $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$.

484 If $A^3 = E$, then $\det A^3 = 1$. Therefore $\det A = 1$. Set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then by computing A^{-1} and A^2, it is easy to conclude that either $A = E$ or $a + d = -1$, $ad - bc = 1$.

485 Either $A = \pm E$ or $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, with $a^2 + bc = \pm 1$.

486 $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = aE + bI$, where $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Thus $I^2 = -E$, and therefore the relation $aE + bI \to a + bi$ is an isomorphism.

487 Set $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = aE + bI + cJ + dK$, where $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then

$$I^2 = J^2 = K^2 = -E,\quad IJ = -JI = K,\quad JK = -KJ = I,\quad KI = -IK = J.$$

This shows that the product of two matrices of the form A + BI + cJ + dK is a matrix of the same form. This is true for sums and differences as

well as products. Thus the matrices form a ring. Moreover if we suppose A $\neq$ 0, then

$$\det A = \begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix} = a^2+b^2+c^2+d^2 \neq 0.$$

Thus every nonzero matrix has an inverse and the equation AB = 0 is only possible if either A = 0 or B = 0. The set of matrices forms not only a ring but a skew field, and is a faithful representation of the algebra of quaternions.

488 $$(a_1E+b_1I+c_1J+d_1K)(a_2E+b_2I+c_2J+d_2K)=(a_1a_2-b_1b_2-c_1c_2-d_1d_2)E+(a_1b_2+b_1a_2+c_1d_2-d_1c_2)I+(a_1c_2-b_1d_2+c_1a_2+d_1b_2)J+(a_1d_2+b_1c_2-c_1b_2+d_1a_2)K.$$

Now take determinants and obtain the following relation

$$\begin{aligned}(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2) &= \\ =(a_1a_2-b_1b_2-c_1c_2-d_1d_2)^2+(a_1b_2+b_1a_2+c_1d_2-d_1c_2)^2 \\ +(a_1c_2-b_1d_2+c_1a_2+d_1b_2)^2+(a_1d_2+b_1c_2-c_1b_2+d_1a_2)^2.\end{aligned}$$

489 The i-th and j-th rows of a matrix are permuted if it is multiplied on the left by the matrix

$$
\begin{array}{rc}
 & \begin{array}{cccccccccc} & & & i & & & j & & & \\ & & & \vdots & & & \vdots & & & \end{array} \\
\begin{array}{r} \\ \\ \\ i \;\cdots \\ \\ \\ \\ j \;\cdots \\ \\ \\ \\ \end{array} &
\left[\begin{array}{cccccccccc}
1 & & & & & & & & & \\
 & \ddots & & & & & & & & \\
 & & 1 & & & & & & & \\
 & & & 0 & \cdots & & 1 & & & \\
 & & & \vdots & 1 & & \vdots & & & \\
 & & & \vdots & & \ddots & \vdots & & & \\
 & & & & & 1 & & & & \\
 & & & 1 & \cdots & & 0 & & & \\
 & & & & & & & 1 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 1
\end{array}\right].
\end{array}
$$

The i-th column of a matrix is augmented by α times the j-th if it is multiplied on the left by one of the matrices

$$
\begin{pmatrix}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & \cdots & \alpha & & \\
 & & & \ddots & \vdots & & \\
 & & & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix},
\quad
\begin{pmatrix}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
 & & \vdots & \ddots & & & \\
 & & \alpha & \cdots & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix}.
$$

The elements of the i-th row are multiplied by the constant α, if the matrix is multiplied on the left by the matrix:

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Multiplication by the same matrices on the right accomplishes the corresponding operations on the columns of the original matrix.

490 The idea of the proof is to carry out elementary transformations of the type described in the preceding problem to bring the matrix A into diagonal form R. According to the preceding problem, R can be expressed in the form $R = U_1 U_2 \ldots U_m A V_1 V_2 \ldots V_k$, where $U_1, \ldots, U_m, V_1, \ldots, V_k$ are the matrices that accomplish the elementary transformations. All of these are non-singular and have inverses. Therefore A = PRQ, where P, Q are non-singular.

491 The proof depends on the factorization in the preceding problem. By problem 489, each of the factors can be written as a product of elementary

matrices, and it is only necessary to establish the assertion for an elementary matrix of each of the three types. Indeed the first type is really superfluous since it can be written as the product of matrices of the other two types. This can be proved either by directly exhibiting a matrix of the first type as a product, or by noting that two rows can be permuted by carrying out the following operations in order: Add the first row to the second, subtract the resulting second row from the first, add the resulting first row to the second and finally multiply the first row by -1. In matrix form this equation reads:

$$E-e_{ii}-e_{kk}+e_{ik}+e_{ki}=(E-2e_{kk})(E+e_{ik})(E-e_{ki})(E+e_{ik}).$$

The last (diagonal) matrix displayed in the solution of problem 489 can be written in the form:

$$E+(\alpha-1)\,e_{11}\ .$$

492 Let $A=P_1R_1Q_1$, $B=P_2R_2Q_2$, where P_1, Q_1, P_2, Q_2 are non-singular matrices and R_1, R_2 have the form diag$\{1,1,\cdots,1,0,\cdots,0\}$ with r_1 and r_2 1's in the main diagonal respectively and zeros elsewhere. Then $AB=P_1R_1Q_1P_2R_2Q_2$, and the rank of AB is the rank of R_1CR_2, $C=Q_1P_2$. Since C is non-singular, the argument can be continued as follows. Every element in the last $n-r_1$ rows and every element in the last $n-r_2$ columns of R_1CR_2 is zero. Since the rank of a matrix cannot be reduced by more than one unit

when a row or a column is removed, it follows that the rank of R_1CR_2 is not less than
$n - (n - r_1) - (n - r_2) = r_1 + r_2 - n$.

493 This is a direct consequence of the fact that all the rows of a matrix of rank 1 are proportional.

494 Problem 492 shows that the rank of the matrix A is either 1 or 0. Therefore

$$A = \begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix}.$$

If we square the matrix we obtain the relation

$$0 = A^2 = (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)\, A,$$

from which it follows that $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0$.

495 Let A be a matrix that satisfies the conditions of the problem, $A \neq \pm \mathbf{E}$. Then one of the matrices $A - \mathbf{E}$, $A + \mathbf{E}$ has rank 1. Set

$$A + E = \begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix} = B.$$

Then $A^2 = E - 2B + B^2 = E + (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 - 2)\, B$, which shows that the condition $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 2$ is necessary and sufficient for the relation $A^2 = \mathbf{E}$. The second case is handled similarly.

496 Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{m1} & \dots & a_{mk} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \dots & b_{1s} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ b_{m1} & \dots & b_{ms} \end{pmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{bmatrix}.$$

Suppose the column C is appended to the matrix (A, B), and suppose that rank B = rank (B, C) i.e. that appending the column C to the matrix B does not increase its rank. Then the following system of equations is compatible.

$$\begin{aligned} b_{11}y_1 + \dots + b_{1s}y_s &= c_1, \\ \dots\dots\dots\dots \\ b_{m1}y_1 + \dots + b_{ms}y_s &= c_m \end{aligned}$$

But this means that the following system is also compatible:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1k}x_k + b_{11}y_1 + \dots + b_{1s}y_s &= c_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + \dots + a_{mk}x_k + b_{m1}y_1 + \dots + b_{ms}y_s &= c_m. \end{aligned}$$

Therefore the matrices (A, B); (A, B, C) have the same rank.

Now let us append the columns of the matrix B to the matrix A one at a time. We have just demonstrated that the rank can be increased by 1 only if the rank of the incomplete matrix B is increased by 1. Therefore rank (A, B) $\leq$ rank A + rank B.

497 Set rank (E + A) = r_1, rank (E - A) = r_2. Since (E + A) + (E - A) = 2E, it follows that $r_1 + r_2 \geq n$. On the other hand (E + A)(E - A) = 0,

from which it follows that $0 \geq r_1 + r_2 - n$. Therefore, $r_1 + r_2 = n$.

498 The rank of the matrix $(\mathbf{E} + A, \mathbf{E} - A)$ is n. Let B be a non-singular sub-matrix, the first r columns of which form a sub-matrix of $\mathbf{E} + A$, and the remaining $n - r$ columns of which form a sub-matrix of $\mathbf{E} - A$. Since the relation $(\mathbf{E} + A)(\mathbf{E} - A) = 0$ holds, we have:

$$(E+A)P = \begin{pmatrix} q_{11} & \dots & q_{1r} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \dots & q_{nr} & 0 & \dots & 0 \end{pmatrix}.$$

$$(E-A)P = \begin{pmatrix} 0 & \dots & 0 & q_{1r+1} & \dots & q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & q_{nr+1} & \dots & q_{nn} \end{pmatrix}.$$

By addition, we obtain:

$$2P = \begin{pmatrix} q_{11} & \dots & q_{1r} & q_{1r+1} & \dots & q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \dots & q_{nr} & q_{nr+1} & \dots & q_{nn} \end{pmatrix}.$$

By subtraction we could have obtained:

$$2AP = \begin{pmatrix} q_{11} & \dots & q_{1r} & -q_{1r+1} & \dots & -q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \dots & q_{nr} & -q_{nr+1} & \dots & -q_{nn} \end{pmatrix} =$$

$$= 2P \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

The assertions above are sufficient to finish the proof.

499 If the elements of the matrix are integers, the value of its determinant must be an integer. If the elements of both A and A^{-1} are integers, then det A must be a unit, ± 1, since $\det A \cdot \det A^{-1} = \det E = 1$. The condition that det A be a unit is obviously also a sufficient condition that all the elements of A^{-1} be integers, as we see for example by applying Cramer's rule to the set of equations used to determine the elements of A as unknowns.

500 Let A be a non-singular matrix with integral elements. The first column must contain nonzero elements. We can furthermore multiply certain rows by -1 and assume that all the elements of the first column are non-negative. I claim that we can perform elementary operations on the rows and obtain a matrix, the first column of which contains α in the first position and zeros elsewhere. The proof is by descent. Operate only with those rows that have nonzero entries in the first column; subtract the row with the smallest such entry from each of the other rows. If there is more than one row with a positive entry in the first column, this process will decrease the value of the largest positive entry and this remark completes the descent.

By further elementary transformations in the last n - 1 rows we can produce zeros in every row below the main diagonal in the second column. Continuing in this manner, we write the original matrix as the product of a unimodular integral matrix by an upper triangular matrix.

The final condition that all the elements in the second factor be non-negative and that the diagonal element in each column be the largest, can be realized by further elementary transformations on the rows. Since the product of the matrices that bring about the elementary transformations needed in the above proof is a unimodular matrix, the assertion is established.

501 Set $A = P_1R_1 = P_2R_2$, where the matrices P_1, R_1; P_2, R_2 satisfy the conditions of problem 500. The relation $P_2^{-1}P_1 = R_2R_1^{-1}$ shows that the matrix $C = P_2^{-1}P_1$ is not only unimodular but also triangular. Set

$$R_1 = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}, \quad R_2 = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ & b_{22} & \dots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix},$$

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ & c_{22} & \dots & c_{2n} \\ & & \ddots & \vdots \\ & & & c_{nn} \end{pmatrix}.$$

The relation $R_2 = CR_1$ shows that $b_{11} = c_{11}a_{11}, \cdots, b_{nn} = a_{nn}c_{nn}$. Thus every element c_{ii} is positive. But $\det c = c_{11}c_{22}\cdots c_{nn} = \pm 1$. Therefore, $c_{11} = c_{22} = \cdots c_{nn} = 1$; $a_{ii} = b_{ii}$.

Next we use the relation $b_{12} = c_{11}a_{12} + c_{12}a_{22} = a_{12} + c_{12}a_{22}$. Solving for c_{12} we obtain $(b_{12} - a_{12})/a_{22}$. But from the relations $0 \leqslant b_{12} < b_{22} = a_{22}$, $0 \leqslant a_{12} < a_{22}$, we obtain $|c_{12}| < 1$. Thus $c_{12} = 0$. Continuing in this way, from one column to another, we find that the matrix relation $CR_1 = R_2$ can be satisfied only if all the non-diagonal elements of the matrix C are zero; $C = \mathbf{E}$. Thus $R_1 = R_2$, $P_1 = P_2$. Thus there is one and only one matrix R in each class.

The number of matrices R with given diagonal elements $d_1, d_2, \cdots, d_n$ is obviously $d_2d_3^2 \ldots d_n^{n-1}$. Therefore the number of matrices R with given determinant k is equal to $F_n(k) = \sum d_2d_3^2 \ldots d_n^{n-1}$, where the sum is extended over all positive integers $d_1, d_2, \cdots, d_n$ such that $d_1d_2 \cdots d_n = k$. If $k = ab$, (a, b) = g.c.d of $a, b = 1$, then every component d_i in the equation $k = d_1d_2 \cdots d_n$ is uniquely factorable into two factors α_i, β_i such that $\alpha_1\alpha_2 \ldots \alpha_n = a$, $\beta_1\beta_2 \ldots \beta_n = b$. Therefore,

$$F_n(k) = \sum_{d_1 d_2 \ldots d_n = k} d_2 d_3^2 \ldots d_n^{n-1} =$$

$$= \sum_{\substack{\alpha_1\alpha_2 \ldots \alpha_n = a \\ \beta_1\beta_2 \ldots \beta_n = b}} \alpha_2\alpha_3^2 \ldots \alpha_n^{n-1}\beta_2\beta_3^2 \ldots \beta_n^{n-1} =$$

$$= \sum_{\alpha_1\alpha_2 \ldots \alpha_n = a} \alpha_2\alpha_3^2 \ldots \alpha_n^{n-1} \cdot \sum_{\beta_1\beta_2 \ldots \beta_n = b} \beta_2\beta_3^2 \ldots \beta_n^{n-1} = F_n(a) \cdot F_n(b).$$

Thus if k has the canonical decomposition $k = p_1^{m_1} \ldots p_s^{m_s}$ it follows that $F_n(k) = F_n(p_1^{m_1}) \ldots F_n(p_s^{m_s})$.

It remains to compute $F_n(p^m)$. To do this we write the formula for $F_n(p^m)$ as the sum of two subsums, in the first of which $d_n = 1$, and in the second of which d_n is divisible by p: $d_n = pd_n'$. Thus we obtain the formula

$$F_n(p^m) = F_{n-1}(p^m) + p^{n-1}F_n(p^{m-1})$$

From this we easily obtain, by mathematical induction, the final result.

$$F_n(p^m) = \frac{(p^{m+1}-1)(p^{m+2}-1)\ldots(p^{m+n-1}-1)}{(p-1)(p^2-1)\ldots(p^{n-1}-1)}.$$

502 The proof of this assertion is similar to that of problem 500, but the details are considerably less complicated. In this problem we are free to perform elementary operations on the rows and columns simultaneously. In this way we obtain a relation $A = U_1U_2 \cdots U_r \, D \, V_1V_2 \cdots V_s$, where

the U_i and the V_j correspond to elementary transformations. The first step is to permute the rows and columns so that the upper left-hand element has the smallest possible absolute value not equal to zero. The next step is to add or subtract the first row to the remaining rows with a view to decreasing the absolute values of the nonzero elements in the first column. At each point in this process further permutations are carried out so that the element with smallest absolute value not equal to zero is always in the upper left-hand corner. Next the first column is added to each of the other columns with the view to decreasing the absolute values of the nonzero elements in the first row, again interspersed with permutations to bring the element of smallest absolute value to the upper left corner. We see that if the original matrix was not the zero matrix, it will be possible to arrange that the upper left-hand element is not zero and the other elements in the first row and also in the first column are zero. The proof is completed by induction, that is by assuming that the theorem is true for all matrices of lower order than that of the given matrix. The final step is to notice that the product of elementary matrices is an integral unimodular matrix.

503 First note the effect of multiplying any integral matrix on the left by the matrices A^m, B^m. If U is any matrix, the product A^mU has the same second row as U, and has a first row obtained by adding m times the second row to the first row. The product B^mU has the same first row as U, and has second row obtained by adding m times the first row to the second row.

We prove the theorem by contradiction. Let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix that cannot be expressed as a product of powers of A, B, and suppose that every matrix $U_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ in which $\max(a_1, c_1) < \max(a, c)$ can be expressed as a product of powers of A, B. Then in fact $\max(a, c) = 1$, since otherwise we could multiply U on the left by positive and negative powers of A, B and thus obtain a matrix expressible as the product of powers of A, B. The proof is now essentially complete; only a small technical calculation remains. If $a = 1$, $c = 0$, then U is already a power of A, since U is known to have determinant 1. The case $a = c = 1$ can be reduced to the previous case if we multiply U on the left by B^{-1}. Finally, the case $a = 0$, $c = 1$ is reduced to the previous case by multiplying U on the left by A.

504 This problem follows from the preceding and two small lemmas. The first lemma asserts that $B = CAC$. The second lemma asserts that a matrix with determinant -1 is equal to the product of another matrix with integral elements and determinant +1 multiplied by the matrix C.

505 Set $\det A = 1$, $A^2 = E$, $A \neq E$. Then by problem 498:

$$A = P \cdot \operatorname{diag}\{1, -1, -1\} \cdot P^{-1}$$

where P is some non-singular matrix. We determine the matrix P so that it is an integral matrix with smallest possible positive determinant. Since the relation $A + E = P \cdot \operatorname{diag}\{2, 0, 0\} \cdot P^{-1}$ holds, the matrix $A + E$ has rank 1 and therefore

$$A+E=\begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix}.$$

Thus by problem 495, $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3=2$. But since the matrix $A + E$ is an integral matrix the numbers $\lambda_1, \lambda_2, \lambda_3$, μ_1, μ_2, μ_3 may be taken as integers.

We now have a set of equations $(A + E) P = P \cdot \{2, 0, 0\}$ that we can solve for the elements of the matrix P. It is not difficult to prove that P can be taken in the form

$$P=\begin{pmatrix}\lambda_1 & 0 & -\delta\\ \lambda_2 & \frac{\mu_3}{\delta} & u\mu_1\\ \lambda_3 & -\frac{\mu_2}{\delta} & v\mu_1\end{pmatrix},$$

where δ is greatest common divisor of μ_2, μ_3; and u, v are integers such that $u\mu_2+v\mu_3=\delta$. P has determinant 2.

By problem 500, P = QR, where Q is unimodular, and R is one of seven possible triangular matrices of determinant 2.

Thus $Q^{-1}AQ$ is equal to one of the seven matrices $R \cdot \mathrm{diag}\{1, -1, -1\} \cdot R^{-1}$. Only three of these matrices are really different, and two of these three differ only by a unimodular transformation. The two matrices remaining are those given in the problem.

506 a) $\begin{pmatrix}9 & 3\\ 10 & 3\end{pmatrix}$; b) $\begin{pmatrix}10\\ 8\end{pmatrix}$; c) $\begin{pmatrix}2 & 4 & 6\\ 1 & 2 & 3\\ 3 & 6 & 9\end{pmatrix}$; d) 13.

507 45.

508 The following identity of Euler is obtained:

$$\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)=(aa_1+bb_1+cc_1)^2+ \\ +(a_1b_2-a_2b_1)^2+(a_1c_2-a_2c_1)^2+(b_1c_2-b_2c_1)^2.$$

509 In the product the minor on the rows with indices $i_1, i_2, \cdots, i_m$ and columns $k_1, k_2, \cdots, k_m$ is the determinant of the product of two matrices. The first of these matrices consists of the rows of the first matrix with indices i, and the second of these matrices consists of the columns of the second factor with indices k. Therefore this minor is equal to the sum of all possible products of m-th order minors on rows i of the first matrix by corresponding minors on columns k of the second matrix.

510 A diagonal minor of the matrix A'A is equal to the sum of the squares of all possible minors of the same order of the matrix A. The sum runs over just the minors from the columns that involve the same indices that appear in the columns of the minor in question in the matrix A'A. Therefore it is non-negative.

511 If all principal k-th order minors of the matrix A'A are zero, then by problem 510, all k-th order minors of k are zero. Therefore the rank of the matrix A, and the rank of the matrix A'A, is less than k.

512 The sum of all the diagonal k-th order minors of either matrix A'A, AA' is equal to the sum of the squares of all k-th order minors of the matrix A.

513 One multiplies the matrix $\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$

by its transpose, and uses the result on the determinant of the product of two matrices.

514 The result is obtained by using the theorem on the determinant of a product:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \cdot \begin{pmatrix} a_1' & b_1' \\ a_2' & b_2' \\ \cdot & \cdot \\ a_n' & b_n' \end{pmatrix}.$$

515 This problem is a consequence of problem 513. Equality can occur only if the rank of the matrix

$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$ is less than 2, that is if the

numbers $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$ are proportional.

516 This follows from problem 514. Equality can occur only if the numbers $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$ are proportional.

517 Suppose the matrix B has m columns and the matrix C has k columns. By Laplace's theorem, det A = $\sum B_i C_i$, where B_i runs through all possible m-th order minors of B, and C_i is the algebraic cofactor of the matrix B_i. Obviously C_i is a submatrix of C. By the Schwarz-Buniakovsky inequality of problem 515,

$(\det A)^2 \leq \sum (\det B_i)^2 \cdot \sum (\det C_i)^2$. But $\sum(\det B_i)^2 = \det B'B$.

518 Set

$$B=\begin{pmatrix} b_{11} & \dots & b_{1m} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{nm} \end{pmatrix}, \quad C=\begin{pmatrix} c_{11} & \dots & c_{1k} \\ \cdot & \cdot & \cdot \\ c_{n1} & \dots & c_{nk} \end{pmatrix}; \quad A=(B, C).$$

The inequality is trivial in the case $m + k > n$, and follows from problem 517 if $m + k = n$. The remaining case, $m + k < n$ is treated as follows. First suppose that, for arbitrary j, s, the equality

$$\sum_{i=1}^{n} b_{ij}c_{is}=0$$

holds. Then

$$A'A = \begin{pmatrix} B'B & 0 \\ 0 & C'C \end{pmatrix} \text{ and therefore}$$

$\det A'A = \det B'B \cdot \det C'C$.

Without loss of generality we can assume that $\operatorname{rank} A = m + k$, because if this is not true the inequality is trivial.

We must append a matrix D to A so that the matrix (A, D) is square and so that D has rank $n - m - k$. We also demand that the inner product of any column of D by any column of A be zero.

This can be accomplished as follows. First obtain the square matrix $\tilde{\varepsilon}' = (A, \tilde{D}')$ choosing $\tilde{D}'$ so that $\tilde{\varepsilon}'$ is non-singular. Next, replace each element of $\tilde{D}'$ by its algebraic cofactor in the matrix $\tilde{\varepsilon}'$. It should be clear that the rank of the matrix D so obtained is equal to the number $n - m - k$ of its columns, since D is a submatrix of the matrix whose elements are the algebraic cofactors of the matrix $\tilde{\varepsilon}'$, and the latter differs from the matrix $\tilde{\varepsilon}'^{-1}$ by the constant factor $\det \tilde{\varepsilon}$. Thus we have shown how to obtain the matrix D needed in the preceding paragraph.

Set $P = (A, D)$; $Q = (C, D)$. By problem 517 we have $\det P'P \le \det B'B \cdot \det Q'Q$. But $\det P'P = \det A'A \cdot \det D'D$; $\det Q'Q = \det C'C \cdot \det D'D$. Since the factor $\det D'D$ is positive, we can factor it out and obtain the result

$$\det A'A \le \det B'B \cdot \det C'C.$$

519 The result follows if we apply problem 518 to the matrix A'.

520 The determinant of the matrix A^*A is the sum of the squares of the moduli of all m-th order minors of A, where m is the number of columns of the matrix A.

521 The solution follows the method used to solve problems 517, 518. For a square matrix one uses Laplace's expansion and the Schwarz-Buniakovsky

inequality. For a rectangular matrix one appends further columns to form a square matrix in such a way that the inner product of each of the appended rows by any row of A is 0. (Note: The inner product of two rows is given by the formula $a_1\overline{b}_1 + a_2\overline{b}_2 + \cdots + a_m\overline{b}_m$.)

522 In problem 521, let B be one of the columns of A. Apply problem 521 several times and obtain:

$$\det(A^*A) = |\det A|^2 \leq \sum_{i=1}^{n} |a_{i1}|^2 \cdot \sum_{i=1}^{n} |a_{i2}|^2 \cdots \sum_{i=1}^{n} |a_{in}|^2 \leqslant n^n M^{2n}$$

thus

$$|\det A| \leq n^{\frac{n}{2}} M^n.$$

523 Border the given matrix by a column of n + 1 elements on the left, all equal to M/2, and a row of n zeros above. Let Δ_1 be the determinant of the new matrix, Δ be the determinant of the original matrix. Then $\Delta = 2\Delta_1/M$. Now subtract the first column of the enlarged matrix from each of the other columns. In the matrix so obtained no element has absolute value exceeding M/2. The assertion now follows by applying the result of problem 522.

524 Let $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. The matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \dots & \varepsilon^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{n-1} & \dots & \varepsilon^{(n-1)^2} \end{bmatrix}$$

has determinant $n^{n/2}$.

525 For $n = 2^m$, an n-th order matrix can be constructed as a Kronecker product of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ with itself m times. For m = 2 the matrix obtained is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Since the inner product of every two rows of such a matrix is 0 one finds:

$$A'A = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & n \end{pmatrix}; \ \det(A'A) = n^n; \ |\det A| = n^{n/2}.$$

If we multiply every element of the matrix A by the constant M, we have the obvious equality:

$$|\det(MA)| = M^n n^{n/2}.$$

526 To see that the absolute value of the determinant can be increased (or at least not decreased) by replacing the element in question by +1 or -1, it is sufficient to examine the expansion of the determinant by minors of a row in which the element in question stands. In this expansion, if the cofactor of the element in question has the same sign as the value of the determinant, the element may be replaced by +1; in the opposite case by -1.

By multiplying appropriate rows by -1 if necessary, we can assume that all the elements of the first column are +1. Except for the first element we can also assume that all the elements in the first row are -1. Now add the first column to each of the remaining columns, and expand the determinant of the matrix by minors of the first row. Since every element beyond the first column in the second, third, fourth,$\cdots$, n-th row is divisible by 2 (being equal to zero or 2), the determinant of the matrix will be divisible by 2^{n-1}.

The proof above shows in fact that the determinant of an $n \times n$ matrix of 1's and -1's must be divisible by 2^{n-1}. Possible values of this determinant are therefore of the form $2^{n-1}N$, where N is the determinant of an $n-1 \times n-1$ matrix of 0's and 1's. The complete set of values that N may have is not known except in the following cases.

$n-1$	N
1	0 or 1
2	0 or 1
3	0, 1, or 2
4	0, 1, 2, or 3
5	0, 1, 2, 3, 4, or 5

527 4 for $n = 3$; 48 for $n = 5$.

528 The result is trivial if A is a singular matrix. Therefore let A be nonsingular and A' be its transpose; let Δ be the determinant; let A_1 be the obverse of A. Then $A_1 = \Delta\, CA'^{-1}C$, where $C = \operatorname{diag}\{-1,1,-1,\cdots\}$. This is clear from the formula for the elements of the inverse of the matrix. Therefore

$$\det A_1 = \Delta^{n-1};\ (A_1)_1 = \Delta^{n-1}C(A'_1)^{-1}C = \Delta^{n-1}\cdot\Delta^{-1}A = \Delta^{n-2}A.$$

529 We use the following notation. Let the minor of the obverse matrix A_1 involve the rows $i_1 < i_2 < \cdots < i_m$ and columns $k_1 < k_2 < \cdots < k_m$. Let the numbers of the remaining rows be $i_{m+1} < i_{m+2} < \cdots < i_n$; and the numbers of the remaining columns be $k_{m+1} < k_{m+2} < \cdots < k_n$. The product of the determinant Δ of the matrix A by the determinant of the minor being considered satisfies the following relations:

$$\Delta \cdot \begin{vmatrix} \Delta_{i_1 k_1} & \cdots & \Delta_{i_1 k_m} \\ \cdot & \cdots & \cdot \\ \Delta_{i_m k_1} & \cdots & \Delta_{i_m k_m} \end{vmatrix}$$

$$= (-1)^{i_1 + \cdots + i_m + k_1 + \cdots + k_m} \Delta \cdot \begin{vmatrix} A_{i_1 k_1} & \cdots & A_{i_m k_1} \\ \cdot & \cdots & \cdot \\ A_{i_1 k_m} & \cdots & A_{i_m k_m} \end{vmatrix}$$

$$= \begin{vmatrix} A_{i_1 k_1} & \cdots & A_{i_m k_1} & A_{i_{m+1} k_1} & \cdots & A_{i_n k_1} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ A_{i_1 k_m} & \cdots & A_{i_m k_m} & A_{i_{m+1} k_m} & \cdots & A_{i_n k_m} \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{vmatrix} \cdot \begin{vmatrix} a_{i_1 k_1} & \cdots & a_{i_1 k_m} & \cdots & a_{i_1 k_n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{i_m k_1} & \cdots & a_{i_m k_m} & \cdots & a_{i_m k_n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{i_n k_1} & \cdots & a_{i_n k_m} & \cdots & a_{i_n k_n} \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & & & & \\ & \ddots & & & \\ & & \Delta & & \\ a_{i_{m+1} k_1} & \cdots & a_{i_{m+1} k_{m+1}} & \cdots & a_{i_{m+1} k_n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{i_n k_1} & \cdots & a_{i_n k_{m+1}} & \cdots & a_{i_n k_n} \end{vmatrix}$$

$$= \Delta^m \cdot \begin{vmatrix} a_{i_{m+1} k_{m+1}} & \cdots & a_{i_{m+1} k_n} \\ \cdot & \cdots & \cdot \\ a_{i_n k_{m+1}} & \cdots & a_{i_n k_n} \end{vmatrix}$$

This relation proves the assertion.

530, 531 These results follow from the theorem on the determinant of the product of two rectangular matrices.

532 The correct ordering is the lexicographic one. This means that the sequence $i_1 < i_2 < \cdots < i_m$ precedes the sequence $j_1 < j_2 < \cdots < j_m$ if the first non-zero difference $i_1 - j_1, i_2 - j_2 \cdots$ is positive. Then every minor of a triagonal matrix has determinant zero if its column index precedes its row index in the above ordering.

533 Exercises 531, 491 indicate that it is sufficient to establish the theorem for a triagonal matrix. By Exercise 532, if A is a triagonal matrix, then

$$\det A_{(m)} = \prod_{i_1 < i_2 < \cdots < i_m} a_{i_1 i_1} a_{i_2 i_2} \cdots a_{i_m i_m} = (\det A)^{C^{n-1}_{m-1}}$$

534 Properties a, b follow immediately from the definition. We establish property c by noting the following formulas, where the notation for elements of the respective matrices is the usual one involving upper and lower case Roman letters. If

$$C = (A' \cdot A'') \times (B' \cdot B''), \quad A' \times B' = G, \quad A'' \times B'' = H,$$

then

$$c_{i_1k_1,\, i_2k_2} = \sum_{i=1}^{n} a'_{i_1 i} a''_{i i_2} \cdot \sum_{k=1}^{m} b'_{k_1 k} b''_{k k_2} = \sum_{i,\, h} a'_{i_1 i} b'_{k_1 k} a''_{i i_2} b''_{k k_2}$$
$$= \sum_{i,\, k} g_{i_1k_1,\, ik} h_{ik,\, i_2k_2}.$$

Thus C = G · H, which establishes the assertion.

535 A verbal proof depends on the fact that the rows and columns of the Kronecker product $A \times B$ can

be permuted to give a matrix which obviously has the determinant asserted.

Another proof depends on the relation

$$A \times B = (A \times E_m) \cdot (E_n \times B).$$

But the determinant of the first factor $A \times E_m$ is the same as the determinant of the matrix

$$\mathrm{diag}\{A, A, \cdots, A\},$$

as is seen by permuting the rows of columns. This remark is sufficient to prove the assertion.

536 In the matrix C_{ik}, the element in the α-th row and β-th column is

$$c_{(i-1)m+\alpha,\,(k-1)m+\beta} = \sum_{s=1}^{mn} a_{(i-1)m+\alpha,s}\, b_{s,\,(k-1)m+\beta}$$
$$= \sum_{j=1}^{n} \sum_{v=1}^{m} a_{(i-1)m+\alpha,\,(j-1)m+v}\, b_{(j-1)m+v,\,(k-1)m+\beta}.$$

Now examine the above formula. The inner sum in the last term is the element of the α-th row and β-th column of the matrix $A_{ij}B_{jk}$. This proves that

$$C_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk}.$$

537 The theorem is trivial for n = 1. We prove the theorem by induction, suppose that it has been proved for a matrix of order n - 1 and use the induction hypothesis to establish the theorem for matrix of order n. Here "order" refers to the number of boxes.

First suppose that the sub-matrix A_{11} is non-singular:

$$C = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}.$$

Let D be given by the formula

$$D = \begin{vmatrix} E, -A_{11}^{-1}A_{12}, \dots, -A_{11}^{-1}A_{1n} & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{vmatrix},$$

and form the product $C' = CD$:

$$C' = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A'_{22} & \dots & A'_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A'_{n2} & \dots & A'_{nn} \end{pmatrix},$$

where $A'_{ik} = A_{ik} - A_{i1}A_{11}^{-1}A_{1k}$.

The explicitly written sub-matrices in the displays for C, D, C' are commutative. This being the case it is easy to check that the formula for the determinant of a product of two matrices remains valid when one considers only formal determinants of "matrices," the elements of which are the displayed sub-matrices.

The formal determinant of the matrix D is E^n; the actual determinant of D is one.

Thus

$$\det C = \det C' = (\det A_{11}) \cdot \det \begin{bmatrix} A'_{22} & \ldots & A'_{2n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ A'_{n2} & \ldots & A'_{nn} \end{bmatrix}.$$

Let B be the formal determinant. Then we have $B = A_{11} \cdot B'$, where B' is the formal determinant of the matrix

$$\begin{pmatrix} A'_{22} & \ldots & A'_{2n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ A'_{n2} & \ldots & A'_{nn} \end{pmatrix}.$$

The induction hypothesis however shows that

$$\det B' = \det \begin{bmatrix} A'_{22} & \ldots & A'_{2n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ A'_{n2} & \ldots & A'_{nn} \end{bmatrix},$$

and therefore $\det B = \det A_{11} \cdot \det B' = \det C$, which establishes the assertion.

If the matrix A_{11} has zero determinant, the proof can be modified by considering the matrix

$$C(\lambda) = \begin{pmatrix} A_{11} + \lambda E_m & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \cdot\ \cdot\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix}$$

and by denoting its formal determinant by $B(\lambda)$.

Note that, for almost all values of λ, the relation $\det C(\lambda) = \det B(\lambda)$ holds, and that $\det(A_{11} + \lambda E_m) = \lambda^m + \cdots \neq 0$, also holds. These relations could not be valid unless the relation $\det C = \det B$ were valid. Thus the assertion is completely established.

The theorem in question remains valid when the elements of the sub-matrices belong to an arbitrary ring. The theorem is due to Ingraham. See also a recent paper by Afriat.
[Ingraham, M.H. A Note on Determinants, Bull. Amer. Math. Soc. 43 (1937), 579-580; and Afriat, S.N. Composite Matrices, Q. J. Math., Oxford Ser(2) 5, (1954), 81-98.]

CHAPTER V - SOLUTIONS

POLYNOMIALS AND RATIONAL FUNCTIONS OF A SINGLE INDETERMINATE

538 a) $2x^6 - 7x^5 + 6x^4 - 3x^3 - x^2 - 2x + 1$;
b) $x^6 + x^5 - 3x^4 - 4x^3 + x^2 + 3x + 1$.

539 a) Quotient $2x^2 + 3x + 11$, remainder $25x - 5$.
b) Quotient $\frac{3x-7}{9}$, remainder $\frac{-26x-2}{9}$.

540 $p = -q^2 - 1$, $m = q$.

541 1) $q = p - 1$, $m = 0$; 2) $q = 1$, $m = \pm\sqrt{2-p}$.

542 $(-1)^n \frac{(x-1)(x-2)\dots(x-n)}{1\cdot 2\cdot 3\dots n}$.

543 a) $(x-1)(x^3 - x^2 + 3x - 3) + 5$;
b) $(x+3)(2x^4 - 6x^3 + 13x^2 - 39x + 109) - 327$:
c) $(x+1+i)[4x^2 - (3+4i)x + (-1+7i)] + 8 - 6i$;
d) $(x-1+2i)[x^2 - 2ix - 5 - 2i] - 9 + 8i$.

544 a) 136; b) $-1 -44i$.

545 a) $(x+1)^4 - 2(x+1)^3 - 3(x+1)^2 + 4(x+1) + 1$;
b) $(x-1)^5 + 5(x-1)^4 + 10(x-1)^3 + 10(x-1)^2 + 5(x-1) + 1$;
c) $(x-2)^4 - 18(x-2) + 38$;
d) $(x+i)^4 - 2i(x+i)^3 - (1+i)(x+i)^2 - 5(x+i) + 7 + 5i$;
e) $(x+1-2i)^4 - (x+1-2i)^3 + 2(x+1-2i) + 1$.

546 a) $\frac{1}{(x-2)^2}+\frac{6}{(x-2)^3}+\frac{11}{(x-2)^4}+\frac{7}{(x-2)^5}$;

b) $\frac{1}{x+1}-\frac{4}{(x+1)^2}+\frac{4}{(x+1)^3}+\frac{2}{(x+1)^5}$.

547 a) $x^4+11x^3+45x^2+81x+55$;

b) $x^4-4x^3+6x^2+2x+8$.

548 a) $f(2)=18$, $f'(2)=48$, $f''(2)=124$, $f'''(2)=216$, $f^{IV}(2)=240$, $f^{V}(2)=120$;

b) $f(1+2i)=-12-2i$, $f'(1+2i)=-16+8i$, $f''(1+2i)=-8+30i$, $f'''(1+2i)=24+30i$, $f^{IV}(1+2i)=24$.

549 a) 3; b) 4.

550 $a = -5$.

551 $A = 3$, $B = -4$.

552 $A = n$, $B = -(n + 1)$.

555 A necessary and sufficient condition that $(x - 1)^{k+1}$ divide $f(x)$ is that $f(1) = a_0 + a_1 + \cdots + a_n = 0$, and that $f'(x)$ be divisible by $(x - 1)^k$. We now observe that the polynomial $f_1(x) = nf(x) - x f'(x)$ will be divisible by $(x - 1)^k$ but not by a higher power of $x - 1$. One treats $f_1(x)$ as a polynomial of formal degree n, and repeats the same argument k times.

556 a is a $k + 3$-fold root, where k is the multiplicity of a as a root of $f'''(x)$.

557 $3125b^2+108a^5=0,\ a\neq 0.$

558 $b=9a^2,\ 1728a^5+c^2=0.$

559 The only multiple root of $x^{n-m-1}[nx^m+(n-m)a]$ is 0.

560 Let d = (m,n) be the greatest common divisor of m and n: $m = dm_1$, $n = dn_1$. Then the result follows from

$$(-1)^{n_1}(n_1-m_1)^{n_1-m_1}m_1^{m_1}a^{n_1}=b^{m_1}n_1^{n_1}.$$

562 If x is a nonzero k - 1-fold root of the polynomial

$$a_1x^{p_1}+a_2x^{p_2}+\dots+a_kx^{p_k}$$

then the following conditions hold:

$$\begin{aligned}
a_1x^{p_1}+{}&a_2x^{p_2}+\dots+a_kx^{p_k}=0,\\
p_1a_1x^{p_1}+{}&p_2a_2x^{p_2}+\dots+p_ka_kx^{p_k}=0,\\
p_1^2a_1x^{p_1}+{}&p_2^2a_2x^{p_1}+\dots+p_k^2a_kx^{p_k}=0,\\
&\dots\dots\dots\dots\dots\dots\\
p_1^{k-2}a_1x^{p_1}+{}&p_2^{k-2}a_2x^{p_2}+\dots+p_k^{k-2}a_kx^{p_k}=0,
\end{aligned}$$

This shows that the numbers $a_1x^{p_1}, a_2x^{p_2}, \dots, a_kx^{p_k}$ are proportional to the algebraic cofactors of the elements of the last row of the Vandermonde matrix

$$\Delta=\begin{bmatrix}1 & 1 & 1 & \dots & 1\\ p_1 & p_2 & p_3 & \dots & p_k\\ \dots & \dots & \dots & \dots & \dots\\ p_1^{k-1} & p_2^{k-1} & p_3^{k-1} & \dots & p_k^{k-1}\end{bmatrix}.$$

It is easy to check that

$$\frac{\det \Delta}{\det \Delta_1} = \prod_{s \neq l} (p_l - p_s) = \varphi'(p_l).$$

This shows that the numbers $a_l x^{p_l}$ are inversely proportional to $\varphi'(p_l)$; thus

$$a_1 x^{p_1} \varphi'(p_1) = a_2 x^{p_2} \varphi'(p_2) = \ldots = a_k x^{p_k} \varphi'(p_k).$$

The converse is established by inverting the order of argument at each step.

563 If f(x) is divisible by f'(x), then the quotient must be a first degree polynomial with highest coefficient 1/n, where n is the degree of f(x). Thus $n f(x) = (x - x_0) f'(x)$. Continued differentiation gives

$$(n-1) f'(x) = (x - x_0) f''(x), \cdots,$$

thus

$$f(x) = \frac{(x - x_0)^n}{n!} f^{(n)}(x) = a_0 (x - x_0)^n.$$

The converse is obvious.

564 A multiple root of the polynomial

$f(x) = 1 + \frac{x}{1} + \ldots + \frac{x^n}{n!}$ will be a root of its derivative also:

$$f'(x) = 1 + \frac{x}{1} + \ldots + \frac{x^{n-1}}{(n-1)!} = f(x) - \frac{x^n}{n!}.$$

Thus if $f(x_0) = f'(x_0)$, then $x_0 = 0$. But 0 is not a root of f(x).

565 If $f(x)=(x-x_0)^k f_1(x)$, where $f_1(x)$ is a rational function that is not zero for $x = x_0$, then successive differentiation gives

$$f(x_0)=f'(x_0)= \dots =f^{(k-1)}(x_0)=0, \quad f^{(k)}(x_0)\neq 0.$$

Conversely if $f(x_0)=f'(x_0)= \dots =f^{(k-1)}(x_0)=0$, $f^{(k)}(x_0)\neq 0$, then $f(x)=(x-x_0)^k f_1(x)$, $f_1(x_0)\neq 0$. For if the relation

$$f(x)=(x-x_0)^m q(x), \quad q(x_0)\neq 0$$

were valid for some $m \neq k$, then the number of successive derivatives that vanished at $x = x_0$ would be greater or less than k.

566 The function

$$g(x)=\frac{\psi(x)}{w(x)}=f(x)-f(x_0)-\frac{f'(x_0)}{1}(x-x_0)- \dots \\ \dots -\frac{f^{(n)}(x)}{n!}(x-x_0)^n$$

satisfies the conditions

$$g(x_0)=g'(x_0)= \dots =g^{(n)}(x_0)=0.$$

Therefore, $\psi(x)=(x-x_0)^{n+1}F(x)$, where $F(x)$ is a polynomial, which proves the assertion.

567 If $f_1(x)f_2(x_0)-f_2(x)f_1(x_0)$ is not identically equal to 0, then we can suppose that $f_1(x_0) \neq 0$. Consider the rational function $\frac{f_2(x)}{f_1(x)}-\frac{f_2(x_0)}{f_1(x_0)}$.

It has x_0 for a root and is not identically zero. The multiplicity is one greater than the multiplicity of x_0 as a root of the derivative. The latter is

equal to $\dfrac{f_1(x)f_2'(x)-f_2(x)f_1'(x)}{[f_1(x)]^2}$. The assertion follows from this fact.

568 Let x_0 be a k-fold root of $[f'(x)]^2-f(x)f''(x)$. Then $f(x_0) \neq 0$, for x_0 cannot be a root of both f(x) and f'(x). By assumption x_0 is a k + 1 fold root of the polynomial $f(x)f'(x_0)-f(x_0)f'(x)$, and the degree of this polynomial does not exceed n. Therefore, $k + 1 \leq n$, $k \leq n - 1$.

569 The polynomial $f(x)f'(x_0)-f(x_0)f'(x)$ must have x_0 as an n-fold root, that is must be of the form $A(x-x_0)^n$, where A is a constant. Set $z = x - x_0$, and expand in powers of z:

$$(a_0+a_1z+a_2z^2+\ldots+a_nz^n)a_1- \\ -(a_1+2a_2z+3a_3z^2+\ldots+na_nz^{n-1})a_0=Az^n,$$

where

$$a_0 = f(x_0) \neq 0.$$

Thus $a_2=\dfrac{a_1^2}{2a_0}$, $a_3=\dfrac{a_1^3}{a_0^2 3!}$, ..., $a_n=\dfrac{a_1^n}{a_0^{n-1}n!}$.

Setting $\dfrac{a_1}{a_0}=\alpha$, we obtain

$$f(x)=a_0\left[1+\frac{\alpha(x-x_0)}{1}+\frac{\alpha^2(x-x_0)^2}{1\cdot 2}+\ldots+\frac{\alpha^n(x-x_0)^n}{n!}\right].$$

570 For example, $\delta = 1/21$ (1/20 does not suffice).

571 For example, $\delta = 1/25$ (1/24 does not suffice).

572 For example, M = 6.

573 For example,

$$a)\ x=\rho i,\ 0<\rho<\sqrt[3]{\frac{4}{3}};$$

$$b)\ x=\rho,\ 0<\rho<\sqrt[3]{\frac{4}{3}}.$$

574 For example, a) $x=1-\rho,\ 0<\rho<\frac{1}{8}$;

b) $x=1+\rho\left(\cos\frac{(2k-1)\pi}{4}+i\sin\frac{(2k-1)\pi}{4}\right),\ \rho<\sqrt[4]{8}$;

c) $x=1+\rho i,\ \rho<\frac{1}{\sqrt{3}}$.

575 Expand the polynomial f(z) in powers of h = z - i to obtain

$$f(z)=(2-i)\left[1+(1-i)h^3-\frac{4+2i}{5}h^4+\frac{1+3i}{5}h^5\right].$$

If we substitute h = a(1 - i), we obtain

$$f(z)=(2-i)\left[1-4a^3+4a^3\left(\frac{4+2i}{5}a-\frac{4+2i}{5}a^2\right)\right],$$

so that

$$|f(z)|\leqslant\sqrt{5}\left(|1-4a^3|+4a^3\frac{1}{4}\sqrt{\frac{4}{5}}\right)<\sqrt{5}$$

for

$$0<a<\frac{1}{2}.$$

576 We write the polynomial in the form
$f(z)=f(z_0)\{1+r(\cos\varphi+i\sin\varphi)(z-z_0)^k[1+(z-z_0)\psi(z)]\}$,
set $z-z_0=\rho(\cos\theta+i\sin\theta)$, choose $\theta=\frac{2m\pi-\varphi}{k}$

and choose ρ small enough so that $|(z-z_0)\psi(z)|<1$. Then

$$|f(z)|=|f(z_0)|\,|1+r\rho^k+r\rho^k(z-z_0)\psi(z)|>f(z_0).$$

577 The proof is similar to that for polynomials and uses Taylor's formula for rational functions, problem 566. Write this formula so that it includes the first term after $f(x_0)$ with non-zero coefficient.

578 Let M be the greatest lower bound for $|f(z)|$ in the given region of the z-plane. It can be proved that there is a point z_0 in this plane, in every neighborhood of which $|f(z)|$ has M as greatest lower bound, by continually subdividing the region. The highest possible power of $z - z_0$ is to be cancelled from numerator and denominator. Call the numerator and denominator in the result $\varphi(z)$, $\psi(z)$ respectively: $f(z) = \varphi(z)/\psi(z)$. Then $\psi(z_0) \neq 0$, since otherwise the least upper bound of $|f(z)|$ could not exist in sufficiently small neighborhoods of z_0. Therefore $f(z)$ is continuous at $z = z_0$, from which it follows that $|f(z_0)| = M$, as it was to be proved.

579 A non-constant rational function does not necessarily have infinite modulus for large values of the independent variable.

580 The hypothesis gives the relations

$$f(a) \neq 0, \quad f'(a) = \ldots = f^{(k-1)}(a) = 0, \quad f^{(k)}(a) \neq 0$$

and by Taylor's formula, we have

$$f(z) = f(a) + \frac{f^{(k)}(a)}{k!}(z-a)^k [1 + \varphi(z)], \quad \varphi(a) = 0.$$

Now set

$$\frac{1}{f(a)}\cdot\frac{f^{(k)}(a)}{k!}=r(\cos\varphi+i\sin\varphi),\quad z-a=\rho(\cos\theta+i\sin\theta).$$

Suppose ρ to be chosen small enough so that

$|\varphi(z)|<1,\ r\rho^k<1$. Then

$$|f(z)|=|f(a)|\cdot|1+r\rho^k[\cos(\varphi+k\theta)+i\sin(\varphi+k\theta)]+r\rho^k\lambda|,$$

where $|\lambda| < 1$. The following relations are clearly valid.

A. $|f(z)| < |f(a)|$ for $\theta = \dfrac{(2m-1)\pi-\varphi}{k}$

B. $|f(z)| > |f(a)|$ for $\theta = \dfrac{2m\pi-\varphi}{k}$

Therefore the value of $|f(z)| - |f(a)|$ changes sign $2k$ times as θ increases from $\{(\pi - \varphi)/k\}$ to $\{(\pi - \varphi)/k\} + 2\pi$. But $|f(z)| - |f(a)|$ is a continuous function of θ and must therefore reduce to 0 $2k$ times, as was to be shown.

581 By the method used in the preceding exercise we can show that if ρ is sufficiently small, then as θ increases by 2π, and $z = \rho(\cos\theta + i\sin\theta)$, the quantities $\mathrm{Re}\{f(z) - f(a)\}$, $\mathrm{Im}\{f(z) - f(a)\}$ change sign $2k$ times. We set $f^{(k)}(a)/k! = r(\cos\varphi + i\sin\varphi)$ and have, by Taylor's formula,

$$f(z)-f(a)=r\rho^k[\cos(\varphi+k\theta)+i\sin(\varphi+k\theta)][1+\varphi(z)],$$

$\varphi(a) = 0$. Now set $\varphi(z)=\varphi_1(z)+i\varphi_2(z)$ and choose ρ so that $|\varphi(z)| < 1$. Then

$$\mathrm{Re}(f(z))-\mathrm{Re}(f(a))=r\rho^k[\cos(\varphi+k\theta)(1+\varphi_1(z)) - \sin(\varphi+k\theta)\varphi_2(z)];$$

$$\operatorname{Im}(f(z)) - \operatorname{Im}(f(a)) = r\rho^k [\sin(\varphi + k\theta)(1 + \varphi_1(z)) + \cos(\varphi + k\theta)\varphi_2(z)].$$

We now choose θ so that

$$\varphi + k\theta = m\pi, \quad m = 0, 1, 2, \ldots, 2k,$$

and obtain

$$\operatorname{Re}(f(z)) - \operatorname{Re}(f(a)) = r\rho^k(-1)^m(1 + \varepsilon_m),$$

where ε_m is the corresponding value of

$$\varphi_1(z), \ |\varepsilon_m| < 1.$$

This shows that $\operatorname{Re}\{f(z) - f(a)\}$ changes sign 2m times when z traces the circumference $|z - a| = \rho$. In a similar manner we can find that $\operatorname{Im}\{f(z) - f(a)\}$ changes sign 2m times, by putting $\varphi + k\theta = \frac{\pi}{2} + m\pi$, m = 0, 1, $\cdots$, 2k.

582 a) $(x-1)(x-2)(x-3)$;

b) $(x-1-i)(x-1+i)(x+1-i)(x+1+i)$;

c) $$\left(x+1-\sqrt{\frac{\sqrt{2}+1}{2}}-i\sqrt{\frac{\sqrt{2}-1}{2}}\right) \times\left(x+1-\sqrt{\frac{\sqrt{2}+1}{2}}+i\sqrt{\frac{\sqrt{2}-1}{2}}\right) \times\left(x+1+\sqrt{\frac{\sqrt{2}+1}{2}}+i\sqrt{\frac{\sqrt{2}-1}{2}}\right) \times\left(x+1+\sqrt{\frac{\sqrt{2}+1}{2}}-i\sqrt{\frac{\sqrt{2}-1}{2}}\right);$$

d) $(x-\sqrt{3}-\sqrt{2})(x-\sqrt{3}+\sqrt{2})(x+\sqrt{3}-\sqrt{2}) \times (x+\sqrt{3}+\sqrt{2})$.

583 a) $2^{n-1}\prod_{k=1}^{n}\left(x-\cos\frac{(2k-1)\pi}{2n}\right)$;

b) $2\prod_{k=1}^{n}\left(x+\frac{\sin\left(\theta+\frac{(2k-1)\pi}{2n}\right)}{\sin\frac{(2k-1)\pi}{2n}}\right)$; c) $\prod_{k=1}^{m}\left(x-\operatorname{tg}^2\frac{(2k-1)\pi}{4m}\right)$.

584 a) $(x^2+2x+2)(x^2-2x+2)$;

b) $(x^2+3)(x^2+3x+3)(x^2-3x+3)$;

c) $\left(x^2+2x+1+\sqrt{2}-2(x+1)\sqrt{\frac{\sqrt{2}+1}{2}}\right)$

$\times\left(x^2+2x+1+\sqrt{2}+2(x+1)\sqrt{\frac{\sqrt{2}+1}{2}}\right)$;

d) $\prod_{k=0}^{n-1}\left(x^2-2\sqrt[2n]{2}\,x\cos\frac{(8k+1)\pi}{4n}+\sqrt[n]{2}\right)$;

e) $(x^2-x\sqrt{a+2}+1)(x^2+x\sqrt{a+2}+1)$;

f) $\prod_{k=0}^{n-1}\left(x^2-2x\cos\frac{(3k+1)2\pi}{3n}+1\right)$.

585 a) $(x-1)^2(x-2)(x-3)(x-1-i)$

$=x^5-(8+i)x^4+(24+7i)x^3-(34+17i)x^2+(23+17i)x$

$-(6+6i)$;

b) $(x+1)^3(x-3)(x-4)=x^5-4x^4-6x^3+16x^2+29x+12$;

c) $(x-i)^2(x+1+i)=x^3+(1-i)x^2+(1-2i)x-1-i$.

586 $\prod_{k=1}^{n}X_k(x)$.

587 a) $(x-1)^2(x-2)(x-3)(x^2-2x+2)=x^6-9x^5+33x^4-65x^3+74x^2-46x+12$;
b) $(x^2-4x+13)^3=x^6-12x^5+87x^4-376x^3+1131x^2-2028x+2197$;
c) $(x^2+1)^2(x^2+2x+2)=x^6+2x^5+4x^4+4x^3+5x^2+2x+2$.

588 a) $(x-1)^2(x+2)$; b) $(x+1)^2(x^2+1)$; c) $(x-1)^3$.

589 x^d-1, where d = (m,n).

590 x^d+a^d, if m/d, n/d are odd; 1, if either is even; d = (m,n).

591 a) $(x - 1)^2(x + 1)$; b) $(x - 1)^3(x + 1)$;
c) x^d-1, d = (m,n) = g.c.d. of m,n.

592 Set $\lambda_0 = u(x_0)/v(x_0)$, and factor f(x) into linear factors: $f(x)=(x-\lambda_0)(x-\lambda_1)\ldots(x-\lambda_{k-1})$.

Then, since $\lambda_j \neq \lambda_0$, we find

$$f\left(\frac{u(x)}{v(x)}\right)=\frac{1}{[v(x)]^k}(u(x)-\lambda_0 v(x))\ldots(u(x)-\lambda_{k-1}v(x)).$$

Now note that since $u(x_0)-\lambda_j v(x_0)=v(x_0)(\lambda_0-\lambda_j)\neq 0$, the hypotheses of the theorem show that $u(x)-\lambda_0 v(x)$ has x_0 as a k-fold root, $k > 1$. Therefore, $u'(x)-\lambda_0 v'(x)$ has x_0 as a k - 1-fold root. Thus,

$$f\left(\frac{u'(x)}{v'(x)}\right)=\frac{1}{[v'(x)]^k}(u'(x)-\lambda_0 v'(x))\ldots(u'(x)-\lambda_{k-1}v'(x)).$$

It is obvious that for $j \neq 0$, the factors $u'(x)-\lambda_j v'(x)$, are all nonzero. Therefore, $f\left(\frac{u'(x)}{v'(x)}\right)$ has x_0 as a k - 1-fold root, as was to be shown.

593 If ω is a root of $x^2 + x + 1$, then $\omega^3 = 1$. Therefore $w^{3m} + w^{3n+1} + w^{3p+2} = 1 + w + w^2 = 0$.

594 If λ is a root of $x^2 - x + 1$, then $\lambda^3 = -1$. Therefore

$$\lambda^{3m} - \lambda^{3n+1} + \lambda^{3p+2} = (-1)^m - (-1)^n \lambda + (-1)^p \lambda^2$$
$$= (-1)^m - (-1)^p + \lambda [(-1)^p - (-1)^n].$$

The last relation shows that the result can be zero only if $(-1)^m = (-1)^p = (-1)^n$, that is if m, n, p are all even or all odd.

595 $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$. The two factors are relatively prime, and $x^2 + x + 1$ always divides $x^{3m} + x^{3n+1} + x^{3p+2}$ by problem 593. It remains to determine when $x^2 - x + 1$ is also a divisor. The method used to solve the preceding problem gives

$$(-1)^m + (-1)^n \lambda + (-1)^p \lambda^2 = (-1)^m - (-1)^p + \lambda [(-1)^n + (-1)^p].$$

Thus we must have $(-1)^m = (-1)^p = -(-1)^n$. Thus the numbers m, p, n + 1 must be all even or all odd.

596 If m is not divisible by 3.

597 All roots of the polynomial $x^{k-1} + x^{k-2} + \ldots + 1$ are k-th roots of 1. Therefore the relation

$$\xi^{ka_1} + \xi^{ka_2+1} + \ldots + \xi^{ka_k+k-1} = 1 + \xi + \ldots + \xi^{k-1} = 0$$

holds. Thus divisibility does occur since all the roots of the first polynomial are simple.

598 When a root ω of the polynomial $x^2 + x + 1$ is substituted into the polynomial $f(x) = (1 + x)^m - x^m - 1$, the result obtained is $(1 + \omega)^m - \omega^m - 1$. But $1 + \omega = -\omega^2 = \lambda$ is a primitive 6-th root of 1. Moreover, $\omega = \lambda^2$; thus $f(\omega) = \lambda^m - \lambda^{2m} - 1$.

For

$$\begin{array}{ll} m = 6n & f(w) = -1 \neq 0; \\ m = 6n+1 & f(w) = \lambda - \lambda^2 - 1 = 0; \\ m = 6n+2 & f(w) = \lambda^2 + \lambda - 1 \neq 0; \\ m = 6n+3 & f(w) = -3 \neq 0; \\ m = 6n+4 & f(w) = -\lambda + \lambda^2 - 1 \neq 0; \\ m = 6n+5 & f(w) = -\lambda^2 + \lambda - 1 = 0. \end{array}$$

$f(x)$ is divisible by $x^2 + x + 1$ if $m = 6n + 1$ or $m = 6n + 5$.

599 For $m = 6n + 2$ and $m = 6n + 4$.

600 $f(w) = m(1+w)^{m-1} - mw^{m-1} = m[\lambda^{m-1} - \lambda^{2(m-1)}]$, $f'(w) = 0$ only for $m = 6n + 1$.

601 For $m = 6n + 4$.

602 No, since the first and second derivatives are not simultaneously equal to 0.

603 For $x = k$, $1 \leq k \leq n$, we have

$$f(k) = 1 - \frac{k}{1} + \frac{k(k-1)}{1\cdot 2} - \dots + (-1)^k \frac{k(k-1)\dots 1}{1\cdot 2\dots k} = (1-1)^k = 0.$$

Thus, the polynomial is divisible by $(x - 1)(x - 2)\cdots(x - n)$. Comparing the highest coefficients, we see that

$$f(x) = \frac{(-1)^n}{n!}(x-1)(x-2)\dots(x-n).$$

604 For the case $(m,n) = 1$.

605 If $f(x^n)$ is divisible by $x - 1$, then $f(1) = 0$. Therefore $f(x)$ is divisible by $x - 1$ and $f(x^n)$ is divisible by $x^n - 1$.

606 If $F(x) = f(x^n)$ is divisible by $(x - a)^k$, then $F'(x)=f'(x^n)nx^{n-1}$ is divisible by $(x-a)^{k-1}$. Thus $f'(x^n)$ is divisible by $(x-a)^{k-1}$.
By the same argument, $f''(x^n)$ is divisible by $(x-a)^{k-2}, \ldots,$ and $f^{(k-1)}(x^n)$ is divisible by $x - a$. Thus we find that $f(a^n)=f'(a^n)=\ldots=f^{k-1}(a^n)=0$ Therefore $f(x)$ is divisible by $(x-a^n)^k$; $f(x^n)$ is divisible by $(x^n-a^n)^k$.

607 If $F(x)=f_1(x^3)+xf_2(x^3)$ is divisible by $x^2 + x + 1$, then $F(w)=f_1(1)+wf_2(1)=0$ where ω is a root of $x^2 + x + 1$. Thus $F(w^2)=f_1(1)+w^2f_2(1)=0$, and finally $f_1(1)=f_2(1) = 0$.

608 The polynomial $f(x)$ has no real root of odd multiplicity since otherwise it would change sign. Therefore $f(x) = [f_1(x)]^2f_2(x)$, where $f_2(x)$ is a polynomial with no real roots. The complex roots of the polynomial f_2 can be classified into two sets, those in one set being complex conjugates of those in the other. Using the linear factors corresponding to the roots in the two sets, we find that $f_2(x)$ is a product of two complex conjugate polynomials:

$$\psi_1(x)+i\psi_2(x) \quad , \quad \psi_1(x)-i\psi_2(x).$$

Therefore

$$f_2(x)=\psi_1^2(x)+\psi_2^2(x) \quad , \quad f(x)=(f_1\psi_1)^2+(f_1\psi_2)^2.$$

609 a) $-x_1, -x_2, \ldots, -x_n$; b) $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$; c) $x_1-a, x_2-a, \ldots, x_n-a$; d) $bx_1, bx_2, \ldots, bx_n$.

610 One of the roots must be $-p/2$. This gives the relation $84 = 4pq - p^3$.

611 $x_1 = 1/6$, $x_2 = 1/2$, $x_3 = -1/3$.

612 $a^3 - 4ab + 8c = 0$.

613 The relation among the roots must be valid for arbitrary α. Setting $\alpha = -a/4$, we obtain the transformed equation

$$y^4+a'y^3+b'y^2+c'y+d'=0, \quad a'=0, \quad a'^3-4a'b'+8c'=0,$$

thus $c' = 0$.

614 $a^2d = c^2$.

615 Division by x^2 gives $x^2+\frac{d}{x^2}+a\left(x+\frac{c}{ax}\right)+b=0.$

Now set $x + \frac{c}{ax} = z$, and obtain

$$x^2+\frac{d}{x^2}=x^2+\frac{c}{a^2x^2}=z^2-2\frac{c}{a},$$

therefore the original equation can be written as a quadratic in z: $z^2 + az + b - 2\frac{c}{a} = 0$. It

is easy to find x after solving for z (generalized reciprocal equation).

616 a) $x = 1 \pm \sqrt{3},\ 1 \pm i\sqrt{2}$; b) $x = 1 \pm 2i,\ -2 \pm i$;
c) $x = \frac{-1 \pm \sqrt{5}}{2},\ \frac{-1 \pm i\sqrt{11}}{2}$; d) $x = 1 \pm \sqrt{3},\ \frac{-3 \pm \sqrt{17}}{2}$.

617 $\lambda = \pm 6$.

618 1) $b = c = 0$, a arbitrary;
2) $a = -1$, $b = -1$, $c = 1$.

619 1) $a = b = c = 0$; 2) $a = 1$, $b = -2$, $c = 0$;
3) $a = 1$, $b = -1$, $c = -1$; 4) $b = \lambda$, $a = -\frac{1}{\lambda}$, $c = \frac{2 - \lambda^2}{\lambda}$, where $\lambda^3 - 2\lambda + 2 = 0$.

620 $\lambda = -3$.

621 $q^3 + pq + q = 0$.

622 $a_1^2 - 2a_2$.

623 $x_i = -\frac{a_1}{n} + \frac{2i - n - 1}{2}h,\quad i = 1, 2, \ldots, n$, where

$$h = \frac{1}{n}\sqrt{\frac{12(n-1)a_1^2 - 24na_2}{n^2 - 1}}.$$

624 If the roots lie in arithmetic progression, then the formula of the preceding exercise gives:

a) $-1/2$, $1/2$, $3/2$ which do satisfy the given equation;

b) $-5/2$, $-3/2$, $-1/2$, $1/2$ which do not satisfy the given equation.

625 Let $y = Ax + B$ be the equation of the given line. Then the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = Ax + B$ must lie in arithmetic progression. By problem 623 we can write them

$$x_i = -\frac{a}{4} + \frac{2i-5}{2}h, \quad i = 1, 2, 3, 4,$$

where

$$h = \frac{1}{2}\sqrt{\frac{9a^2 - 24b}{15}} = \frac{1}{2}\sqrt{\frac{3a^2 - 8b}{5}}.$$

Moreover

$$A - c = x_1x_4(x_2 + x_3) + x_2x_3(x_1 + x_4) =$$
$$= -\left(\frac{a^2}{16} - \frac{9}{4}h^2\right)\frac{a}{2} - \left(\frac{a^2}{16} - \frac{1}{4}h^2\right)\frac{a}{2} = \frac{a^3 - 4ab}{8},$$
$$d - B = x_1x_2x_3x_4 = \frac{1}{1600}(36b - 11a^2)(4b + a^2).$$

Thus

$$A = \frac{a^3 - 4ab + 8c}{8}, \quad B = d - \frac{1}{1600}(36b - 11a^2)(4b + a^2).$$

The points of intersection will be real and distinct if $3a^2 - 8b > 0$, that is if the second derivative $2(6x^2 + 3ax + b)$ changes sign as x traverses the real axis.

626 $x^4 - ax^2 + 1 = 0$, where $a = \frac{\alpha^4 + 1}{\alpha^2}$.

627 $(x^2 - x + 1)^3 - a(x^2 - x)^2 = 0, \quad a = \frac{(\alpha^2 - \alpha + 1)^3}{(\alpha^2 - \alpha)^2}$.

628
$$f'(x_i) = (x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n);$$
$$f''(x_i) = 2[(x_i - x_2)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)$$
$$+ (x_i - x_1)(x_i - x_3)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)$$
$$+ \dots + (x_i - x_1)(x_i - x_2)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots$$
$$\dots(x_i - x_{n-1})] = 2f'(x_i)\sum_{\substack{s=1 \\ (s \neq i)}}^{n} \frac{1}{x_i - x_s} \quad (\text{if } x_s \neq x_i).$$

629 Follows immediately from problem 628.

630 Set $x_i = x_1 + (i-1)h$. Then

$$f'(x_i) = (-1)^{n-i}(i-1)!\,(n-i)!\,h^{n-1}.$$

631 a) $x+1$; b) x^2+1; c) x^3+1; d) x^2-2x+2; e) x^3-x+1; f) $x+3$; g) x^2+x+1; h) $x^2-2x\sqrt{2}-1$; i) $x+2$; j) 1; k) $2x^2+x-1$; l) x^2+x+1.

632 a) $(-x-1)f_1(x)+(x+2)f_2(x)=x^2-2$;
b) $-f_1(x)+(x+1)f_2(x)=x^3+1$;
c) $(3-x)f_1(x)+(x^2-4x+4)f_2(x)=x^2+5$;
d) $(1-x^2)f_1(x)+(x^3+2x^2-x-1)f_2(x)=x^3+2$;
e) $(-x^2+x+1)f_1(x)+(x^3+2x^2-5x-4)f_2(x)=3x+2$;
f) $-\frac{x-1}{3}f_1(x)+\frac{2x^2-2x-3}{3}f_2(x)=x-1$.

633 a) $M_2(x)=x$, $M_1(x)=-3x^2-x+1$;
b) $M_2(x)=-x-1$, $M_1(x)=x^3+x^2-3x-2$;
c) $M_2(x)=\frac{-x^2+3}{2}$, $M_1(x)=\frac{x^4-2x^2-2}{2}$;
d) $M_2(x)=-\frac{2x^2+3x}{6}$, $M_1(x)=\frac{2x^3+5x^2-6}{6}$;
e) $M_2(x)=3x^2+x-1$, $M_1(x)=-3x^3+2x^2+x-2$;
f) $M_2(x)=-x^3-3x^2-4x-2$,
$M_1(x)=x^4+6x^3+14x^2+15x+7$.

634 a) $M_2(x)=\frac{-16x^2+37x+26}{3}$, $M_1(x)=\frac{16x^3-53x^2-37x-23}{3}$;
b) $M_2(x)=4-3x$, $M_1(x)=1+2x+3x^2$;
c) $M_2(x)=35-84x+70x^2-20x^3$,
$M_1(x)=1+4x+10x^2+20x^3$.

635 a) $M_1(x)=9x^2-26x-21$,
$M_2(x)=-9x^3+44x^2-39x-7$;
b) $M_1(x)=3x^3+3x^2-7x+2$,
$M_2(x)=-3x^3-6x^2+x+2$.

636 a) $4x^4-27x^3+66x^2-65x+24$;
b) $-5x^7+13x^6+27x^5-130x^4+75x^3+266x^2-440x+197$.

637
$$N(x)=1+\frac{n}{1}x+\frac{n(n+1)}{1\cdot 2}x^2+\dots$$
$$\dots+\frac{n(n+1)\dots(n+m-2)}{1\cdot 2\dots(m-1)}x^{m-1};$$
$$M(x)=1+\frac{m}{1}(1-x)+\frac{m(m+1)}{1\cdot 2}(1-x)^2+\dots$$
$$\dots+\frac{m(m+1)\dots(m+n-2)}{1\cdot 2\dots(n-1)}(1-x)^{n-1}$$
$$=\frac{(m+1)(m+2)\dots(m+n-1)}{(n-1)!}-\frac{m}{1}\,\frac{(m+2)\dots(m+n-1)}{(n-2)!}x$$
$$+\frac{m(m+1)}{1\cdot 2}\,\frac{(m+3)\dots(m+n-1)}{(n-3)!}x^2-\dots$$
$$\dots+(-1)^{n-1}\frac{m(m+1)\dots(m+n-2)}{(n-1)!}x^{n-1}.$$

638 1.

639 a) $(x+1)^4(x-2)^2$ b) $(x+1)^4(x-4)$;
c) $(x-1)^3(x+3)^2(x-3)$; d) $(x-2)(x^2-2x+2)^2$;
e) $(x^3-x^2-x-2)^2$; f) $(x^2+1)^2(x-1)^3$;
g) $(x^4+x^3+2x^2+x+1)^2$.

640 a) $f(x)=x+1+\frac{1}{24}x(x-1)(x-2)(x-3)$;
b) $f(x)=-x^4+4x^3-x^2-7x+5$;
c) $f(x)=1+\frac{2}{5}(x-1)-\frac{1}{105}(x-1)(4x-9)+$
$+\frac{1}{945}(x-1)(4x-9)(x-4)$,
$$f(2)=1\frac{389}{945}=1,4116\ \dots\qquad (\sqrt{2}=1,4142\ \dots);$$
d) $f(x)=x^3-9x^2+21x-8$.

641 a) $y=-\frac{1}{3}(x-2)(x-3)(x-4)$

$$+\frac{1}{2}(x-1)(x-3)(x-4)-2(x-1)(x-2)(x-4)$$

$$+\frac{1}{2}(x-1)(x-2)(x-3)=-\frac{4}{3}x^3+10x^2-\frac{65}{3}x+15;$$

b) $y=\frac{1}{2}[5-(1-i)x-x^2-(1+i)x^3]$.

642 $f(x)=\frac{n+1}{2}-\frac{1}{2}\sum_{k=1}^{n-1}\left(1-i\operatorname{ctg}\frac{k\pi}{n}\right)x^k.$

Solution.

$$f(x)=\sum_{s=0}^{n-1}\frac{(s+1)(x^n-1)}{\left(x-\varepsilon_s\right)n\varepsilon_s^{n-1}}$$

$$=\frac{1}{n}\sum_{s=0}^{n-1}\frac{(s+1)(1-x^n)}{\left(1-x\varepsilon_s^{-1}\right)}=\frac{1}{n}\sum_{s=0}^{n-1}\sum_{k=0}^{n-1}(s+1)x^k\varepsilon_1^{-ks}$$

$$=\frac{1}{n}\sum_{k=0}^{n-1}x^k\sum_{s=0}^{n-1}(s+1)\varepsilon_1^{-ks}=\frac{1}{n}\sum_{s=0}^{n-1}(s+1)$$

$$+\frac{1}{n}\sum_{k=1}^{n-1}x^k\sum_{s=0}^{n-1}(s+1)\varepsilon_k^{-s}=\frac{n+1}{2}-\sum_{k=1}^{n-1}\frac{x^k}{1-\varepsilon_k^{-1}}$$

$$=\frac{n+1}{2}-\frac{1}{2}\sum_{k=1}^{n-1}\left(1-i\operatorname{ctg}\frac{k\pi}{n}\right)x^k.$$

643 $f(x)=\sum_{k=1}^{n}\frac{y_k(x^n-1)}{\left(x-\varepsilon_k\right)n\varepsilon_k^{n-1}}=\frac{1}{n}\sum_{k=1}^{n}\frac{y_k(1-x^n)}{1-x\varepsilon_k^{-1}},\quad f(0)=\frac{1}{n}\sum_{k=1}^{n}y_k.$

644 Set $\varphi(x)=(x-x_1)(x-x_2)\dots(x-x_n)$. Let f(x) be an arbitrary polynomial of degree n - 1 or less; let it assume the values $y_1, y_2, \cdots, y_n$ for $x = x_1, x_2, \cdots, x_n$. Then

$$f(x_0)=\frac{y_1+y_2+\dots+y_n}{n}=\sum_{k=1}^{n}\frac{y_k\varphi(x_0)}{\varphi'(x_i)(x_0-x_i)}.$$

Since $y_1, y_2, \cdots, y_n$ are arbitrary, then

$$\frac{\varphi(x_0)}{\varphi'(x_i)(x_0 - x_i)} = \frac{1}{n}.$$

Consider the polynomial

$$F(x) = n[\varphi(x_0) - \varphi(x)] - (x_0 - x)\varphi'(x).$$

Its degree is $< n$, and it takes the value 0 for $x = x_1, x_2, \cdots, x_n$. Thus, $F(x) = 0$. Expand $\varphi(x)$ in powers of $(x - x_0)$:

$$\varphi(x) = \sum_{k=0}^{n} c_k (x - x_0)^k.$$

We find $\sum_{k=1}^{n} (n-k) c_k (x - x_0)^k = 0$. Thus $c_1 = c_2 = \ldots = c_{n-1} = 0;$

$$\varphi(x) = (x - x_0)^n + c_0, \quad x_i = x_0 + \sqrt[n]{-c_0}.$$

645 $$x^s = \sum_{i=1}^{n} \frac{x_i^s \varphi(x)}{(x - x_i)\varphi'(x_i)}.$$

Comparing coefficients (of x^{n-1}) we obtain

$$\sum_{i=1}^{n} \frac{x_i^s}{\varphi'(x_i)} = 0.$$

646 $$x^{n-1} = \sum_{i=1}^{n} \frac{x_i^{n-1}\varphi(x)}{(x - x_i)\varphi'(x_i)}.$$

Comparing coefficients (of x^{n-1}) we obtain

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\varphi'(x_i)} = 1.$$

647 $a_i = \frac{1}{\Delta} \sum_{k=1}^{n} y_k \Delta_{ki}$, where

$$\Delta = \det \begin{vmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & \ldots & x_n^{n-1} \end{vmatrix},$$

Δ_{ki} is the algebraic cofactor of the element in the k-th row and i + 1-th column in the matrix above.

$$f(x)=\sum_{i=0}^{n-1} a_i x^i=\frac{1}{\Delta}\sum_{k=1}^{n} y_k \sum_{i=0}^{n-1} \Delta_{ki}x^i=\sum_{k=1}^{n} y_k \frac{\Delta_k}{\Delta},$$

where Δ_k is the determinant of the matrix obtained from the matrix above by replacing the k-th row by $1, x, \cdots, x^{n-1}$.

Expanding the determinants, Δ_k, Δ as Vandermonde determinants gives

$$\frac{\Delta_k}{\Delta}=\frac{(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}=\frac{\varphi(x)}{(x-x_k)\varphi'(x)},$$

where $\varphi(x)=(x-x_1)(x-x_2)\dots(x-x_n)$.

Thus $f(x)=\sum \frac{y_k\varphi(x)}{(x-x_k)\varphi'(x_k)}$, which was to be proved.

648 $$f(x)=1+\frac{x}{1!}+\frac{x(x-1)}{2!}+\dots+\frac{x(x-1)\dots(x-n+1)}{n!}.$$

649 $$f(x)=1+\frac{(a-1)x}{1}+\frac{(a-1)^2x(x-1)}{1\cdot 2}+\dots$$
$$\dots+\frac{(a-1)^n x(x-1)\dots(x-n+1)}{n!}.$$

650 $$f(x)=1-\frac{2x}{1}+\frac{2x(2x-2)}{1\cdot 2}+\dots$$
$$\dots+\frac{2x(2x-2)\dots(2x-4n+2)}{(2n)!}.$$

651 $$f(x)=1-\frac{x-1}{2!}+\frac{(x-1)(x-2)}{3!}-\dots$$
$$\dots+(-1)^n\frac{(x-1)(x-2)\dots(x-n+1)}{n!}=\frac{n!-(1-x)(2-x)\dots(n-x)}{n!\,x}.$$

652 $f(x)=\frac{\varphi(a)-\varphi(x)}{\varphi(a)(x-a)}$, where $\varphi(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$.

653 We write f(x) in the form

$$f(x)=A_0+A_1\frac{x-m}{1}+A_2\frac{(x-m)(x-m-1)}{1\cdot 2}+\ldots$$
$$\ldots+A_n\frac{(x-m)(x-m-1)\ldots(x-m-n+1)}{n!},$$

where $m, m+1,\cdots,m+n$ are the integral values of x for which the polynomial f(x) takes on integral values.

Replacing x by $m, m+1,\cdots,m+n$, we obtain the following set of equations for determining $A_0, A_1,\cdots,A_n$:

$$A_0=f(m),$$
$$A_k=f(m+k)-A_0-\frac{k}{1}A_1-\frac{k(k-1)}{1\cdot 2}A_2-\ldots-kA_{k-1}.$$
$$k=1, 2, \ldots, n,$$

From these equations it is clear that all coefficients A_k have integral values. When x has any integral value, each term of f(x) is the product of a binomial coefficient by an integer A_k, and therefore is itself an integer. Therefore, f(x) takes on integral values for integral x, as was to be proved.

654 The polynomial $F(x) = f(x^2)$ of degree 2n takes on integral values for the $2n + 1$ values $x = -n, -(n-1),\cdots,-1, 0, 1,\cdots,n$, and by the preceding problem takes on integral values for all integral values of x.

655 a) $\frac{1}{12(x-1)}-\frac{4}{3(x+2)}+\frac{9}{4(x+3)}$;

b) $\frac{1}{6(x-1)}+\frac{1}{2(x-2)}-\frac{1}{2(x-3)}+\frac{1}{6(x-4)}$;

c) $\frac{2}{x-1}+\frac{-2+i}{2(x-i)}+\frac{-2-i}{2(x+i)}$;

d) $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{i}{4(x-i)}+\frac{i}{4(x+i)}$;

e) $\frac{1}{3}\left(\frac{1}{x-1}+\frac{\varepsilon}{x-\varepsilon}+\frac{\varepsilon^2}{x-\varepsilon^2}\right)$, $\varepsilon=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$;

f) $-\frac{1}{16}\left(\frac{1+i}{x-1-i}+\frac{1-i}{x-1+i}+\frac{-1+i}{x+1-i}+\frac{-1-i}{x+1+i}\right)$;

g) $\frac{1}{n}\sum_{k=0}^{n-1}\frac{\varepsilon_k}{x-\varepsilon_k}$, $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$;

h) $-\frac{1}{n}\sum_{k=1}^{n}\frac{\eta_k}{x-\eta_k}$, $\eta_k=\cos\frac{(2k-1)\pi}{n}+i\sin\frac{(2k-1)\pi}{n}$;

i) $\sum_{k=0}^{n}\frac{C_n^k(-1)^{n-k}}{x-k}$; j) $\sum_{k=-n}^{n}\frac{(-1)^{n-k}C_{2n}^{n+k}}{x-k}$;

k) $\frac{1}{n}\sum_{k=1}^{n}\frac{(-1)^{k-1}\sin\frac{2k-1}{2n}\pi}{x-\cos\frac{2k-1}{2n}\pi}$.

656 a) $\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}$;

b) $\frac{1}{8(x-2)}-\frac{1}{8(x+2)}+\frac{1}{2(x^2+4)}$;

c) $\frac{1}{8}\frac{x+2}{x^2+2x+2}-\frac{1}{8}\frac{x-2}{x^2-2x+2}$;

d) $\frac{1}{18}\left(\frac{1}{x^2+3x+3}+\frac{1}{x^2-3x+3}-\frac{2}{x^2+3}\right)$;

e) $\frac{1}{2n+1}\left[\frac{1}{x-1}+2\sum_{k=1}^{n}\frac{x\cos\frac{2k(m+1)\pi}{2n+1}-\cos\frac{2km\pi}{2n+1}}{x^2-2x\cos\frac{2k\pi}{2n+1}+1}\right]$;

f) $$\frac{(-1)^m}{2n+1}\left[\frac{1}{x+1}+2\sum_{k=1}^{n}\frac{x\cos\frac{2k(m+1)\pi}{2n+1}+\cos\frac{2km\pi}{2n+1}}{x^2+2x\cos\frac{2k\pi}{2n+1}+1}\right];$$

g) $$\frac{1}{2n}\left[\frac{1}{x-1}-\frac{1}{x+1}+2\sum_{k=1}^{n-1}\frac{x\cos\frac{k\pi}{n}-1}{x^2-2x\cos\frac{k\pi}{n}+1}\right];$$

h) $$\frac{1}{n}\sum_{k=1}^{n}\frac{\cos\frac{(2k-1)m\pi}{n}-x\cos\frac{(2k-1)(2m+1)\pi}{2n}}{x^2-2x\cos\frac{(2k-1)\pi}{2n}+1};$$

i) $$\frac{1}{(n!)^2x}+2\sum_{k=1}^{n}\frac{(-1)^kx}{(n+k)!\,(n-k)!\,(x^2+k^2)}.$$

657 a) $$\frac{1}{4(x-1)^2}-\frac{1}{4(x+1)^2};$$

b) $$\frac{1}{4(x+1)}-\frac{1}{4(x-1)}+\frac{1}{4(x-1)^2}+\frac{1}{4(x+1)^2};$$

c) $$\frac{3}{(x-1)^3}-\frac{4}{(x-1)^2}+\frac{1}{x-1}-\frac{1}{(x+1)^2}-\frac{2}{x+1}+\frac{1}{x-2};$$

d) $$\frac{1}{n^2}\left[\sum_{k=0}^{n-1}\frac{\varepsilon_k^2}{(x-\varepsilon_k)^2}-(n-1)\sum_{k=0}^{n-1}\frac{\varepsilon_k}{(x-\varepsilon_k)}\right],$$

$$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n};$$

e) $$\frac{1}{x^m}+\frac{\frac{n}{1}}{x^{m-1}}+\frac{\frac{n(n+1)}{1\cdot 2}}{x^{m-2}}+\dots+\frac{\frac{n(n+1)\dots(n+m-2)}{1\cdot 2\dots(m-1)}}{x}$$

$$+\frac{1}{(1-x)^n}+\frac{\frac{m}{1}}{(1-x)^{n-2}}+\frac{\frac{m(m+1)}{1\cdot 2}}{(1-x)^{n-2}}+\dots$$

$$\dots+\frac{\frac{m(m+1)\dots(m+n-2)}{1\cdot 2\dots(n-1)}}{1-x};$$

f) $$\frac{1}{(-4a^2)^n}\sum_{k=0}^{n-1}(2a)^{n-k}\frac{n(n+1)\dots(n+k-1)}{k!}$$

$$\times\left[\frac{1}{(a-x)^{n-k}}+\frac{1}{(a+x)^{n-k}}\right];$$

g) $$\frac{1}{(4a^2)^n}\sum_{k=0}^{n-1}(2a)^{n-k}\frac{n(n+1)\dots(n+k-1)}{k!}\times\left[\frac{1}{(a-ix)^{n-k}}+\frac{1}{(a+ix)^{n-k}}\right];$$

h) $$\sum_{k=1}^{n}\frac{g(x_k)}{[f'(x_k)]^2(x-x_k)^2}+\sum_{k=1}^{n}\frac{g'(x_k)f'(x_k)-g(x_k)f''(x_k)}{[f'(x_k)]^3(x-x_k)}.$$

658 a) $-\frac{1}{4(x+1)}+\frac{x-1}{4(x^2+1)}+\frac{x+1}{2(x^2+1)^2}$;

b) $-\frac{1}{x}+\frac{7}{x+1}+\frac{3}{(x+1)^2}-\frac{6x+2}{x^2+x+1}-\frac{3x+2}{(x^2+x+1)^2}$;

c) $$\frac{1}{16(x-1)^2}-\frac{3}{16(x-1)}+\frac{1}{16(x+1)^2}+\frac{3}{16(x+1)}+\frac{1}{4(x^2+1)}+\frac{1}{4(x^2+1)^2};$$

d) $$\frac{1}{4n^2}\left[\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{2n-1}{x-1}+\frac{2n-1}{x+1}\right]+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{\sin^2\frac{k\pi}{n}\left(1-2x\cos\frac{k\pi}{n}\right)}{\left(x^2-2x\cos\frac{k\pi}{n}+1\right)^2}+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{n-\sin^2\frac{k\pi}{n}-\left(n-\frac{1}{2}\right)x\cos\frac{k\pi}{n}}{x^2-2x\cos\frac{k\pi}{n}+1}.$$

659 a) $\frac{\varphi'(x)}{\varphi(x)}$; b) $\frac{x\varphi'(x)-n\varphi(x)}{\varphi(x)}$; c) $\frac{[\varphi'(x)]^2-\varphi(x)\varphi''(x)}{[\varphi(x)]^2}$.

660 a) 9; b) $-\frac{\varphi'(2)}{\varphi(2)}+\frac{\varphi'(1)}{\varphi(1)}=-\frac{17}{5}$; c) 17.

661 $0.51x + 2.04$.

662 $y=\frac{1}{7}[0.55x^2+2.35x+6.98]$.

663 We have

$$q^n f(p/q) = a_0p^n + a_1p^{n-1}q + \dots + a_{n-1}pq^{n-1} + a_nq^n = 0 \ ;$$

thus

$$\frac{a_0p^n}{q} = -(a_1p^{n-1} + \dots + a_{n-1}pq^{n-2} + a_nq^{n-1}),$$
$$\frac{a_nq^n}{p} = -(a_0p^{n-1} + a_1p^{n-1}q + \dots + a_{n-1}q^{n-1}).$$

The right member of each of the last two equations is an integer. The numbers p, q are relatively prime. Therefore, q divides a_0, and p divides a_n.

Now expand f(x) in powers of x - m:

$$f(x) = a_0(x-m)^n + c_1(x-m)^{n-1} + \dots + c_{n-1}(x-m) + c_n.$$

The coefficients $c_1, c_2, \cdots, c_n$ are integers, since m is an integer. Also, $c_n = f(m)$. Now substitute $x = p/q$ in the above relationship:

$$f(p/q) = a_0(p-mq)^n + c_1(p-mq)^{n-1}q + \dots + c_{n-1}(p-mq)q^{n-1} + c_nq^n = 0.$$

Thus we can conclude that $\dfrac{c_nq^n}{p-mq}$ is an integer.

In view of the fact that $\dfrac{p - mq}{q} = \dfrac{p}{q} - m$ is irreducible, the numbers p = mq, q are relatively prime. Therefore $c_n = f(m)$ is divisible by p - mq, which was to be proved.

664 Detailed reasoning for part a) is as follows.

Possible values of p are: 1, -1, 2, -2, 7, -7, 14, -14. The only possible value of q is 1 (we have already taken account of both positive and negative values).

$f(1) = -4$. Therefore $p - 1$ must divide 4. This eliminates the values $p = 1, -2, 7, -7, 14, -14$. Remaining values to be tested are -1, 2. But $f(-1) \neq 0$; $f(2) = 0$. The only rational root is $x_1 = 2$.

b) $x_1 = -3$; c) $x_1 = -2$, $x_2 = 3$; d) $x_1 = -3$, $x_2 = \frac{1}{2}$;
e) $\frac{5}{2}$, $-\frac{3}{4}$; f) 1, -2, 3; g) $\frac{1}{2}$, $-\frac{2}{3}$, $\frac{3}{4}$;
h) No rational root; i) -1, -2, -3, $+4$;
k) $\frac{1}{2}$; l) $x_1 = x_2 = -\frac{1}{2}$; m) $x_1 = x_2 = 1$, $x_3 = x_4 = -3$;
n) $x_1 = 3$, $x_2 = x_3 = x_4 = x_5 = -1$; o) $x_1 = x_2 = x_3 = 2$.

665 By problem 663, p and $p - q$ are both odd. Therefore q is even and cannot be 1.

666 By problem 663, $p = x_1 q = \pm 1$, $p - x_2 q = \pm 1$. Thus $(x_2 - x_1)q = \pm 2$, or 0. The value 0 is impossible, since $q > 0$, $x_2 \neq x_1$. For definiteness, suppose $x_2 > x_1$, $(x_2 - x_1)q = 2$. This would not be possible unless $x_2 - x_1 = 2$ or 1. In the first case, p and q must be related by $p = x_1 q + 1$, $q = \frac{2}{x_2 - x_1}$. Then $p/q = x_1 + 1/q = (x_1 + x_2)/2$.

667 Eisenstein's criterion gives the results:
a) $p = 2$; b) $p = 3$; c) $p = 3$, if one expands in powers of $x - 1$.

668 $$X_p(x)=(x-1)^{p-1}+\frac{p}{1}(x-1)^{p-2}+\frac{p(p-1)}{1\cdot 2}(x-1)^{p-3}+\ldots+p.$$

All the coefficients $C_k=\frac{p(p-1)\ldots(p-k+1)}{1\cdot 2\ldots k}$, $k \le p-1$, are divisible by p, since $k!C_k=p(p-1)\ldots(p-k+1)$ is divisible by p, but k! is prime to p. Thus Eisenstein's criterion applies if p is prime and we expand in powers of x - 1, as we did.

669 We can use Eisenstein's criterion, if we substitute x = y + 1, and pay attention to divisibility by p:

$$X_{p^k}(x)=\varphi(y)=\frac{(y+1)^{p^k}-1}{(y+1)^{p^{k-1}}-1}.$$

The highest coefficient of the polynomial φ is 1. Furthermore the constant term of $\varphi(y)$ is $\varphi(0)=X_{p^k}(1)=p$; this is divisible by p and not by p^2. It remains to show that the other coefficients are divisible by p. The proof will be complete if we show by induction on n that all the coefficients of the polynomial $(y+1)^{p^n}-1$, except the leading one, are divisible by p. This is obviously true for n = 1. Suppose as induction hypothesis that this is true for when the exponent is p^{n-1}, i.e.
$(y+1)^{p^{n-1}}=y^{p^{n-1}}+1+pw_{n-1}(y)$, where $w_{n-1}(y)$

is a polynomial with integer coefficients. Then

$$(y+1)^{p^n}=(y^{p^{n-1}}+1+pw_{n-1}(y))^p=(y^{p^{n-1}}+1)^p+p\psi(y)=$$
$$=y^{p^n}+1+pw_n(y);$$

where $\psi(y)$, $\omega_n(y)$ are polynomials with integral coefficients. Thus,

$$\varphi(y)=\frac{y^{p^k}+pw_k(y)}{y^{p^{k-1}}+pw_{k-1}(y)}$$
$$=y^{p^k-p^{k-1}}+p\,\frac{w_k(y)-y^{p^k-p^{k-1}}w_{k-1}(y)}{y^{p^{k-1}}+pw_{k-1}(y)}=y^{p^k-p^{k-1}}+p\chi(y).$$

The coefficients of the polynomial $\chi(y)$ are integers since $\chi(y)$ is the quotient of a division of two polynomials, and the highest coefficient of the divisor is 1. Therefore all the coefficients of the polynomial $\chi(y)$ except for the leading coefficient are divisible by p, and Eisenstein's criterion applies.

670 To prove the assertion, assume the contrary:

$$f(x)=\varphi(x)\psi(x).$$

If the factors are non-trivial, both factors are polynomials with integral coefficients and degree exceeding 1, since f(x) has no rational root, by hypothesis. Set

$$\varphi(x)=b_0x^k+b_1x^{k-1}+\ldots+b_k,$$
$$\psi(x)=c_0x^m+c_1x^{m-1}+\ldots+c_m,$$

$k \geq 2$, $m \geq 2$, $k+m=n$. Since $b_kc_m = a_n$ is divisible by p and is not divisible by p^2, we can assume that b_k is divisible by p, and c_m is not divisible by p.

Let i be chosen, $i \geq 0$, so that $b_{i+1}, b_{i+2}, \ldots$ are divisible by p, b_i is not divisible by p. Such an i exists, since $a_0 = b_0 c_0$ is not divisible by p. Then $a_{m+i} = b_i c_m + b_{i+1} c_{m-1} + \cdots$ is not divisible by p, since the first term is not divisible by p and the remaining terms are divisible by p. This contradicts the assumption, since $m + i \geq 2$.

671 Suppose that f(x) can be written as a product of non-trivial factors, each of which is an irreducible polynomial with integral coefficients. Let $\varphi(x)$ be an irreducible factor having constant term divisible by p. There must be such a factor since a_n is divisible by p. Let $\psi(x)$ be the complementary factor: $f(x) = \varphi(x)\ \psi(x)$. The proof now proceeds along the lines of the argument for problem 670. Set

$$\varphi(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m,$$
$$\psi(x) = c_0 x^h + c_1 x^{h-1} + \ldots + c_h$$

and suppose that $b_{i+1}, b_{i+2}, \cdots$ are divisible by p, but b_i is prime to p. Such an i exists, since c_h cannot be divisible by p, otherwise $a_n = b_m c_h$ would be divisible by p^2.

Thus $a_{h+i} = b_i c_h + b_{i+1} c_{h-1} + \cdots$ is not divisible by p, from which the relation $h + i \leq k$ follows. Therefore $m \geq m + h + i - k = n + i - k \geq n - k$.

672 a) f(0) = 1, f(1) = -1, f(-1) = -1.

If $f(x)=\varphi(x)\psi(x)$, where the degree of $\varphi(x) \leq 2$, then $\varphi(0) = \pm 1$, $\varphi(1) = \pm 1$, $\varphi(-1) = \pm 1$, that is $\varphi(x)$ takes one of the sets of tabular values shown below:

x	$\varphi(x)$							
-1	1	1	1	1	-1	-1	-1	-1
0	1	-1	-1	1	-1	1	1	-1
1	-1	-1	1	1	1	1	-1	-1

We can consider some of these cases together since the last four cases are the same as the respective negatives of the first four. Moreover the fourth and last cases can be omitted since the corresponding polynomial would be the constant 1. Thus we arrive at the following three possibilities: $\varphi(x)=-(x^2+x-1)$; $\varphi(x)=x^2-x-1$; $\varphi(x)=2x^2-1$.

The following factorization can be verified:

$$f(x) = (x^2 + x - 1)(x^2 - x - 1).$$

b) Irreducible; c) Irreducible;

d) $(x^2 - x - 1)(x^2 - 2)$.

673 A reducible first degree polynomial must have a factor of the first degree that has rational coefficients. Thus the original polynomial must have a rational root.

674 If the polynomial $x^4+ax^3+bx^2+cx+d$ has no rational root, then it must be either irreducible or expressible as the product of two quadratic

polynomials with integral coefficients:

$$x^4+ax^3+bx^2+cx+d=(x^2+\lambda x+m)(x^2+\mu x+n).$$

Comparing coefficients we obtain the following relations:

$$\begin{aligned} mn &= d, \\ \lambda+\mu &= a, \\ n\lambda+m\mu &= c. \end{aligned}$$

If $m \neq n$, then $\lambda=\frac{c-am}{n-m}=\frac{cm-am^2}{d-m^2}$, as the assertion states. If $m = n$, then $c = am$, and $c^2 = a^2d$ (see problem 614). This is the exceptional case mentioned in the problem.

675 Suppose the polynomial is reducible. Then

$$x^5+ax^4+bx^3+cx^2+dx+e$$
$$=(x^2+\lambda x+m)(x^3+\lambda' x^2+\lambda'' x+n)$$

We may assume all explicitly written coefficients to be integers.

Comparing coefficients we obtain the relations:

$$\begin{aligned} mn &= e, \\ n\lambda+m\lambda'' &= d, \\ m+\lambda\lambda'+\lambda'' &= b, \\ n+\lambda\lambda''+m\lambda' &= c. \end{aligned}$$

Thus

$$m\lambda'' - n\lambda' = d - an,$$
$$\lambda(m\lambda'' - n\lambda') + m^2\lambda' - n\lambda'' = cm - bn.$$

Therefore $(d-an)\lambda+m^2\lambda'-n\lambda''=cm-bn.$ Solving this equation together with $\lambda+\lambda'=a$, $n\lambda+m\lambda''=d$, we obtain

$$\lambda=\frac{am^3-cm^2-dn+be}{m^3-n^2+ae-dm},$$

which is what was to be proved.

676 a) $(x^2 - 2x + 3)(x^2 - x - 3)$; b) not factorable;
c) $(x^2 - x - 4)(x^2 + 5x + 3)$;
d) $(x^2 - 2x + 2)(x^3 + 3x + 3)$.

677 If the polynomial $x^4 + px^2 + q$ with rational coefficients is reducible, then it is expressible as a product of a third degree by a first degree polynomial or as a product of two second degree polynomials. But we need not consider the first case, since if x_1 is a root, then $-x_1$ is also a root and the product $(x - x_1)(x + x_1)$ is a second degree factor with rational coefficients.

Set $x^4 + px^2 + q = (x^2 + \lambda_1 x + \mu_1)(x^2 + \lambda_2 x + \mu_2)$.
Then

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0, \\ \lambda_1\mu_2 + \lambda_2\mu_1 &= 0, \\ \mu_1 + \lambda_1\lambda_2 + \mu_2 &= p, \\ \mu_1\mu_2 &= q. \end{aligned}$$

If $\lambda_1 = 0$, then $\lambda_2 = 0$. In this case a necessary and sufficient condition that rational numbers μ_1, μ_2 exist is that the discriminant $p^2 - 4q$ should be the square of a rational number.

The remaining case is $\lambda_1 \neq 0$. Then $\lambda_2 = -\lambda_1$, $\mu_2 = \mu_1$ and thus

$$q = \mu_1^2, \quad 2\mu_1 - p = \lambda_1^2.$$

In summary, the polynomial $x^4 + px^2 + q$ with rational coefficients will be reducible if and only if one of the following two conditions is satisfied.

a) $p^2 - 4q$ is the square of a rational number.

b) q is the square of a rational number μ_1, and $2\mu_1 - p$ is the square of a rational number λ_1.

678 Suppose $x^4+ax^3+bx^2+cx+d=(x^2+p_1x+q_1)(x^2+p_2x+q_2)$. Then, since $p_1 + p_2 = a$, we can write:

$$x^4+ax^3+bx^2+cx+d=\left(x^2+\frac{1}{2}ax+\frac{\lambda}{2}\right)^2-\left(\frac{p_1-p_2}{2}x+\frac{q_1-q_2}{2}\right)^2,$$

where $\lambda = q_1 + q_2$. Thus the auxiliary cubic has the rational root $\lambda = q_1 + q_2$.

679 Suppose $f(x)=\varphi(x)\psi(x)$ where $\varphi(x)$, $\psi(x)$ are polynomials with integral coefficients. Since $f(a_i) = -1$, then either $\varphi(a_i)=1$, $\psi(a_i)=-1$, or $\varphi(a_i)=-1$, $\psi(a_i)=1$. If neither $\varphi(x)$ or $\psi(x)$ is constant, then the degree of $\varphi(x)+\psi(x)$ is less than n; thus $\varphi(x)+\psi(x) \equiv 0$. This leads to the contradiction $f(x) = -[\varphi(x)]^2$. This is indeed a contradiction since the coefficient of the highest power of x in f(x) is positive.

680 If $f(x)=\varphi(x)\psi(x)$, then $\varphi(a_i)=\psi(a_i)=\pm 1$, as in the preceding problem. If neither $\varphi(x)$ or $\psi(x)$ is constant, then $\varphi(x) \equiv \psi(x)$ and thus

$$f(x)=[\varphi(x)]^2.$$

This can be true only if the degree n is even.

Thus the only possibility to consider is

$$(x-a_1)(x-a_2)\dots(x-a_n)+1=[\varphi(x)]^2.$$

Assume without loss of generality the coefficient of the highest power of x in $\varphi(x)$ is +1 (and not

-1). Thus

$$\varphi(x)+1=(x-a_1)(x-a_3)\dots(x-a_{n-1}).$$
$$\varphi(x)-1=(x-a_2)(x-a_4)\dots(x-a_n).$$

(In these last equations the subscripts are a typographical simplification that can be obtained by renumbering the roots $a_1, a_2, \cdots, a_n$.) Finally we have

$$(x-a_1)(x-a_3)\dots(x-a_{n-1})-(x-a_2)(x-a_4)\dots(x-a_n)=2.$$

Suppose $a_1>a_3>\dots>a_{n-1}$. In the equation last obtained we can set $x = a_{2k}$, for $k = 1,2,\cdots,n/2$ And obtain:

$$(a_{2k}-a_1)(a_{2k}-a_3)\dots(a_{2k}-a_{n-1})=2.$$

Thus the number 2 is factored in n/2 different ways into a product of n/2 integral factors. Moreover the factors appear in increasing order. This can occur only if $n/2 = 2$, $2 = -2\,(-1) = 1 \cdot 2$, or also if $n/2 = 1$. These two possibilities correspond precisely to the two factorizations mentioned in the problem.

681 If the n-th degree polynomial f(x) is reducible, $n = 2m$ or $n = 2m + 1$, then f(x) must have one factor $\varphi(x)$ of degree not exceeding m. If f(x) takes the value ± 1 for more than 2m integral values of its argument, then $\varphi(x)$ also takes the value ± 1 for the same values of the argument. Thus $\varphi(x)$ must take at least one of the two values +1, -1 more than m times. This shows that either $\varphi(x) \equiv +1$ or $\varphi(x) \equiv -1$.

682 If $f(x)$ could be expressed as the product $f(x) = \varphi(x)\ \psi(x)$ of non-trivial factors, then neither factor could have a real root since $f(x)$ has no real root. Thus neither factor can change sign as x traverses the real line, and we may suppose $\varphi(x) > 0$, $\psi(x) > 0$ for all real x. Since $f(a_k) = 1$, it follows that $\varphi(a_k) = \psi(a_k) = 1$, $k = 1,2,\cdots,n$. If the degree of either $\varphi(x)$ or $\psi(x)$ were less than n, then the polynomial would have to be identically 1. Thus each of the factors $\varphi(x)$, $\psi(x)$ has degree n. Thus

$$\varphi(x) = 1 + \alpha(x - a_1)\ldots(x - a_n),\ \psi(x) = 1 + \beta(x - a_1)\ldots(x - a_n),$$

where α, β are certain integers. Thus we have shown that $f(x)$ can be represented in the form

$$f(x) = (x - a_1)^2\ldots(x - a_n)^2 + 1 = 1 + (\alpha + \beta)(x - a_1)\ldots(x - a_n) + \alpha\beta(x - a_1)^2\ldots(x - a_n)^2.$$

Comparing the coefficients of x^{2n}, x^n, we obtain the relations: $\alpha\beta = 1$, $\alpha + \beta = 0$; but no integers satisfy these relations, and therefore $f(x)$ is irreducible.

683 Suppose that $f(x)$ takes the value 1 for more than three values of its argument. Then $f(x) - 1$ must have at least four different integral roots, i.e.

$$f(x) - 1 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)\,h(x),$$

where a_1, a_2, a_3, a_4 and the coefficients of the polynomial $h(x)$ are integers. For any particular integral value of x, the product

$(x-a_1)(x-a_2)(x-a_3)(x-a_4)$ is a product of distinct integers. No more than two of these can be 1 or -1; the other two must always be different from ± 1. Therefore the product cannot be a prime number; in particular cannot be -2. Thus, $f(x) - 1 \neq -2$, for any integral value of x. Therefore $f(x) \neq -1$.

684 Suppose that $f(x)=\varphi(x)\psi(x)$. One of the polynomials $\varphi(x)$ has degree $\leq n/2$, and takes one of the values ± 1 for more than n/2 integral values of x. Since $n/2 \geq 6$, $\varphi(x)$ or $-\varphi(x)$ takes the value 1 more than three times, and according to problem 683, cannot take the value of -1 at all. Thus, either $\varphi(x)$ or $-\varphi(x)$ takes the value +1 more than n/2 times. It must therefore be identically 1 and f(x) is irreducible. For $n \geq 8$, the assertion can be established by a different argument.

685 Suppose

$$a[\varphi(x)]^2+b\varphi(x)+1=\psi(x)\,\omega(x).$$

One of the polynomials has degree $\leq n$; $\psi(x)$ takes the value ± 1 for $x = a_1, a_2, \cdots, a_n$, and since $n \geq 7$, all these values of $\psi(x)$ must be of the same sign. Therefore,

$$\psi(x)=\pm 1+\alpha(x-a_1)(x-a_2)\ldots(x-a_n)=\pm 1+\alpha\varphi(x).$$

If $\alpha \neq 0$, then $\omega(x)$ must also have degree n, and $\omega(x) = \pm 1 + \beta\varphi(x)$. But the equality

$$a[\varphi(x)]^2+b\varphi(x)+1=[\pm 1+\alpha\varphi(x)][\pm 1+\beta\varphi(x)]$$

cannot subsist, since by hypothesis the polynomial $ax^2 + bx + 1$ is irreducible.

686 a) $f(x) = a_0 x^n \left(1 + \frac{a_1}{a_0 x} + \ldots + \frac{a_n}{a_0 x^n}\right)$.

Set $\left|\frac{a_k}{a_0}\right| = A$. Then for $|x| > 1$:

$|f(x)| \geqslant |a_0 x^n| \left[1 - \frac{A}{|x| - 1}\right] = |a_0 x^n| \frac{|x| - 1 - A}{|x| - 1}$; this is positive for $|x| > 1 + A$.

b) $\frac{1}{\rho^n} f(x) = a_0 \left(\frac{x}{\rho}\right)^n + \frac{a_1}{\rho}\left(\frac{x}{\rho}\right)^{n-1} + \frac{a_2}{\rho^2}\left(\frac{x}{\rho}\right)^{n-2} + \ldots + \frac{a_n}{\rho^n}$.

By a) we find that all the roots satisfy the relations

$\frac{|x|}{\rho} \leqslant 1 + \max \left|\frac{a_k}{a_0 \rho^k}\right|$, thus $|x| \leqslant \rho + \max \left|\frac{a_k}{a_0 \rho^{k-1}}\right|$.

c) Set $\rho = \max \sqrt[k]{\left|\frac{a_k}{a_0}\right|}$. Then

$$\left|\frac{a_k}{a_0}\right| \leqslant \rho^k, \quad \left|\frac{a_k}{a_0 \rho^{k-1}}\right| \leqslant \rho, \quad \max \left|\frac{a_k}{a_0 \rho^{k-1}}\right| \leqslant \rho.$$

Therefore the moduli of all the roots are bounded by the number

$$\rho + \rho = 2\rho = 2 \max \sqrt[k]{\left|\frac{a_k}{a_0}\right|}.$$

d) Set $\rho = \max \sqrt[k-1]{\left|\frac{a_k}{a_1}\right|}$. Then

$$|a_k| \leqslant |a_1| \rho^{k-1}, \left|\frac{a_k}{a_0 \rho^{k-1}}\right| \leqslant \left|\frac{a_1}{a_0}\right|.$$

Therefore no root has modulus exceeding

$$\rho + \left|\frac{a_1}{a_0}\right| = \left|\frac{a_1}{a_0}\right| + \max \sqrt[k-1]{\left|\frac{a_k}{a_1}\right|}.$$

687 Set $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$,
$\varphi(x)=b_0x^n-b_1x^{n-1}-\dots-b_n$;

$0<b_0\leqslant|a_0|,\ b_1\geqslant|a_1|,\ \dots,\ b_n\geqslant|a_n|$. Obviously, $|f(x)|\geqslant\varphi(|x|)$.

Thus, $\varphi(x)=b_0x^n\left(1-\frac{b_1}{b_0x}-\frac{b_2}{b_0x^2}-\dots-\frac{b_n}{b_0x^n}\right)$.

The factor in parentheses increases from $-\infty$ to 1 as x varies from 0 to ∞.

Thus $\varphi(x)$ has a single positive root ξ and $\varphi(x)>0$ for $x>\xi$. Thus, for $|x|>\xi$, the relation $|f(x)|\geqslant\varphi(|x|)>0$, holds, and therefore no root of f(x) exceeds ξ in modulus.

688 a) Set $A=\max\left|\frac{a_k}{a_0}\right|$. It is obvious that

$$|f(x)|\geqslant|a_0x^n|\left(1-\frac{A}{|x|^r}-\frac{A}{|x|^{r+1}}-\dots-\frac{A}{|x|^n}\right).$$

Thus, for $|x|>1$, we have

$$|f(x)|>|a_0x^n|\left(1-\frac{A}{|x|^{r-1}(|x|-1)}\right)$$
$$=\frac{|a_0x^{n-r+1}|}{|x|-1}[|x|^{r-1}(|x|-1)-A]>\frac{|a_0x|^{n-r+1}}{|x|-1}[(|x|-1)^r-A].$$

For $|x|>1+\sqrt[r]{A}$, $|f(x)|>0$.

b) $\frac{1}{\rho^n}f(x)=a_0\left(\frac{x}{\rho}\right)^n+\frac{a_r}{\rho^r}\left(\frac{x}{\rho}\right)^{n-r}+\dots+\frac{a_n}{\rho^n}$.

By a) all roots of f(x) satisfy the relation

$$\left|\frac{x}{\rho}\right|<1+\sqrt[r]{\max\left|\frac{a_k}{a_0\rho^k}\right|},\text{ so that }|x|<\rho+\sqrt[r]{\max\left|\frac{a_k}{a_0\rho^{k-r}}\right|}.$$

c) Set $\rho=\max\sqrt[k-1]{\left|\frac{a_k}{a_r}\right|}$. Then $|a_k|\leqslant|a_r|\rho^{k-r}$,

and no root of the polynomial exceeds

$$\sqrt[r]{\left|\frac{a_r}{a_0}\right|}+\rho=\sqrt[r]{\left|\frac{a_r}{a_0}\right|}+\max\sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}$$

in modulus.

689 The assertion is obvious for the negative roots of the polynomial. For definiteness suppose $a_0 > 0$, and set $\varphi(x)=a_0x^n-b_1x^{n-1}-b_2x^{n-2}-\ldots-b_n$, where $b_k = 0$ for $a_k > 0$, $b_k = -a_k$ for $a_k < 0$. Then if x is positive, we have

$$f(x) \geq \varphi(x).$$

Thus $\varphi(x)$ has a single non-negative root ξ (see problem 687) and $\varphi(x) > 0$ for $x > \xi$. Thus, for $x > \xi$, $f(x) \geq \varphi(x) > 0$.

690 Follows directly from problems 688, 689, 686 c).

692 Expand f(x) in powers of x - a. For $x \leq a$, we have:

$$f(x)=f(a)+\frac{f'(a)}{1}(x-a)+\frac{f''(a)}{1\cdot 2}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n>0.$$

693 We use problems 690, 692 to obtain an upper bound for the roots. A lower bound can be obtained by considering f(-x):

a) $0<x_i<3$; b) $0<x_i<1$; c) $-11<x_i<11$; d) $-6<x_i<2$.

694 a) $f=x^3-3x-1$, $f_1=x^2-1$, $f_2=2x+1$, $f_3=+1$.

Three real roots in the intervals (-2, -1), (-1, 0), (1, 2).

b) $f=x^3+x^2-2x-1$, $f_1=3x^2+2x-2$, $f_2=2x+1$, $f_3=+1$.

Three real roots in the intervals (-2, -1), (-1, 0), (1, 2).

c) $f=x^3-7x+7,\ f_1=3x^2-7,\ f_2=2x-3,\ f_3=+1.$

Three real roots in the intervals (-4, -3), (1, 3/2), (3/2, 2).

d) $f=x^3-x+5,\ f_1=3x^2-1,\ f_2=2x-15,\ f_3=-1.$

One real root in the interval (-2, -1).

e) $f=x^3+3x-5,\ f_1=x^2+1.$

One real root in the interval (1, 2).

695 a) $f=x^4-12x^2-16x-4,\ f_1=x^3-6x-4,$

$f_2=3x^2+6x+2,\ f_3=x+1,\ f_4=1.$

Four real roots in the intervals (-3, -2), (-2, -1), (-1, 0), (4, 5).

b) $f=x^4-x-1,\ f_1=4x^3-1,\ f_2=3x+4,\ f_3=+1.$

Two real roots in the intervals (-1, 0), (1, 2).

c) $f=2x^4-8x^3+8x^2-1, f_1=x^3-3x^2+2x, f_2=2x^2-4x+1,$
$f_3=x-1, f_4=1.$

Four real roots in the intervals (-1, 0), (0, 1), (1, 2), (2, 3).

d) $f=x^4+x^2-1,\ f_1=2x^3+x,\ f_2=-x^2+2,\ f_3=-x,$
$f_4=-1.$

Two real roots in the intervals (-1, 0), (0, 1).

e) $f=x^4+4x^3-12x+9,\ f_1=x^3+3x^2-3,\ f_2=x^2+3x-4,$
$f_3=-4x+3,\ f_4=1.$

No real roots.

696 a) $f=x^4-2x^3-4x^2+5x+5,\ f_1=4x^3-6x^2-8x+5,$
$f_2=22x^2-22x-45,\ f_3=2x-1,\ f_4=1.$

Four real roots in the intervals (1, 2), (2, 3), (-1, 0), (-2, -1).

b) $f = x^4 - 2x^3 + x^2 - 2x + 1$, $f_1 = 2x^3 - 3x^2 + x - 1$, $f_2 = x^2 + 5x - 3$, $f_3 = -9x + 5$, $f_4 = -1$.

Two real roots in the intervals (0, 1), (1, 2).

c) $f = x^4 - 2x^3 - 3x^2 + 2x + 1$, $f_1 = 2x^3 - 3x^2 - 3x + 1$, $f_2 = 9x^2 - 3x - 5$, $f_3 = 9x + 1$, $f_4 = +1$.

Four real roots in the intervals (-2, -1), (-1, 0), (0, 1), (2, 3).

d) $f = x^4 - x^3 + x^2 - x - 1$, $f_1 = 4x^3 - 3x^2 + 2x - 1$, $f_2 = -5x^2 + 10x + 17$, $f_3 = -8x - 5$, $f_4 = -1$.

Two real roots in the intervals (1, 2), (-1, 0).

e) $f = x^4 - 4x^3 - 4x^2 + 4x + 1$, $f_1 = x^3 - 3x^2 - 2x + 1$, $f_2 = 5x^2 - x - 2$, $f_3 = 18x + 1$, $f_4 = +1$.

Four real roots in the intervals (-2, -1), (-1, 0), (0, 1), (4, 5).

697 a) $f = x^4 - 2x^3 - 7x^2 + 8x + 1$, $f_1 = 2x^3 - 3x^2 - 7x + 4$, $f_2 = 17x^2 - 17x - 8$, $f_3 = 2x - 1$, $f_4 = 1$.

Four real roots in the intervals (-3, -2), (-1, 0), (1, 2), (3, 4).

b) $f = x^4 - 4x^2 + x + 1$, $f_1 = 4x^3 - 8x + 1$, $f_2 = 8x^2 - 3x - 4$, $f_3 = 87x - 28$, $f_4 = +1$.

Four real roots in the intervals (-3, -2), (-1, 0), (0, 1), (1, 2).

c) $f = x^4 - x^3 - x^2 - x + 1$, $f_1 = 4x^3 - 3x^2 - 2x - 1$, $f_2 = 11x^2 + 14x - 15$, $f_3 = -8x + 7$, $f_4 = -1$.

Two real roots in the intervals (0, 1), (1, 2).

d) $f = x^4 - 4x^3 + 8x^2 - 12x + 8$, $f_1 = x^3 - 3x^2 + 4x - 3$, $f_2 = -x^2 + 5x - 5$, $f_3 = -9x + 13$, $f_4 = -1$.

Two real roots $x_1 = 2$, $1 < x_2 < 2$.

e) $f = x^4 - x^3 - 2x + 1$, $f_1 = 4x^3 - 3x^2 - 2$, $f_2 = 3x^2 + 24x - 14$, $f_3 = -56x + 31$, $f_4 = -1$.

Two real roots in the intervals (0,1), (1,2).

698 a) $f=x^4-6x^2-4x+2,\ f_1=x^3-3x-1,$
$f_2=3x^2+3x-2,\ f_3=4x+5,\ f_4=1.$

Four real roots in the intervals (-2, -3/2), (-3/2, -1), (0, 1), (2, 3).

b) $f=4x^4-12x^2+8x-1,\ f_1=2x^3-3x+1,$
$f_2=6x^2-6x+1,\ f_3=2x-1,\ f_4=1.$

Four real roots in the intervals (-3, -2), (0, 1/2), (1/2, 1), (1, 2).

c) $f=3x^4+12x^3+9x^2-1,\ f_1=2x^3+6x^2=3x,$
$f_2=9x^2+9x+2,\ f_3=13x+8,\ f_4=1.$

Four real roots in the intervals (-4, -3), (-1, -2/3), (-2/3, -1/2), (0, 1).

d) $f=x^4-x^3-4x^2+4x+1,\ f_1=4x^3-3x^2-8x+4,$
$f_2=7x^2-8x-4,\ f_3=4x-5,\ f_4=1.$

Four real roots in the intervals (1, 3/2), (3/2, 2), (-2, -1), (-1, 0).

e) $f=9x^4-126x^2-252x-140,\ f_1=x^3-7x-7,$
$f_2=9x^2+27x+20,\ f_3=2x+3,\ f_4=1.$

Four real roots in the intervals (4, 5), (-4/3, -1), (-5/3, -4/3), (-2, -5/3).

699 a) $f=2x^5-10x^3+10x-3,\ f_1=x^4-3x^2+1,$
$f_2=4x^3-8x+3,\ f_3=4x^2+3x-4,\ f_4=x,\ f_5=1.$

Five real roots in the intervals (-2, -3/2), (-3/2, -1), (0, 1/2), (1/2, 1), (1, 2).

b) $f=x^6-3x^5-3x^4+11x^3-3x^2-3x+1,\ f_1=2x^5-5x^4$
$-4x^3+11x^2-2x-1,\ f_2=3x^4-6x^3-x^2+4x-1,\ f_3=4x^3$
$-6x^2+1,\ f_4=26x^2-26x+5,\ f_5=2x-1,\ f_6=1.$

Six real roots in the intervals (-2, -1), (-1, 0), (0, 1/2), (1/2, 1), (1, 2), (2, 3).

c) $f = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, $f_1 = 5x^4 + 4x^3 - 12x^2 - 6x + 3$, $f_2 = 4x^3 + 3x^2 - 6x - 2$, $f_3 = 3x^2 + 2x - 2$, $f_4 = 2x + 1$, $f_5 = 1$.

Five real roots in the intervals (-2, -3/2), (-3/2, -1), (-1, 0), (0, 1), (1, 2).

d) $f = x^5 - 5x^3 - 10x^2 + 2$, $f_1 = x^4 - 3x^2 - 4x$, $f_2 = x^3 + 3x^2 - 1$, $f_3 = -2x^2 + x + 1$, $f_4 = -3x - 1$, $f_5 = -1$.

Three real roots in the intervals (-1, 0), (0, 1), (2, 3).

700 a) $f = x^4 + 4x^2 - 1$, $f_1 = x$, $f_2 = 1$.

Two real roots in the intervals (-1, 0), (0, 1).

b) $f = x^4 - 2x^3 + 3x^2 - 9x + 1$, $f_1 = 2x - 3$, $f_2 = 1$.

Two real roots in the intervals (0, 1), (2, 3).

c) $f = x^4 - 2x^3 + 2x^2 - 6x + 1$, $f_1 = 2x - 3$, $f_2 = 1$.

Two real roots in the intervals (0, 1), (2, 3).

d) $f = x^5 + 5x^4 + 10x^2 - 5x - 3$, $f_1 = x^2 + 4x - 1$, $f_2 = 5x - 1$, $f_3 = 1$.

Three real roots in the intervals (0, 1), (-1, 0), (-6, -5).

701 The Sturm sequence consists of the polynomials $x^3 + px + q$, $3x^2 + p$, $-2px - 3q$, $-4p^3 - 27q^2$. If $-4p^3 - 27q^2 > 0$, then $p < 0$. In each polynomial in the Sturm sequence the highest coefficient is positive and thus all the roots of $x^3 + px + q$ are real. If $-4p^3 - 27q^2 < 0$, then the sign of p does not matter; the Sturm sequence has two sign changes for $x = -\infty$; one change for $x = +\infty$. In this case $x^3 + px + q$ has one real root.

702 The polynomials

$$x^n + px + q,\ nx^{n-1} + p,\ -(n-1)px - nq,\ -p - n\left(\frac{-nq}{(n-1)p}\right)^{n-1}$$

form the Sturm sequence.

If n is odd, the sign of the last polynomial is the same as the sign of $\Delta = -(n-1)^{n-1}p^n - n^n q^{n-1}$. If $\Delta > 0$, then we must have $p < 0$. In this case the polynomial has three real roots. If $\Delta < 0$, then the sign of p does not matter, and the polynomial has one real root.

If n is even, the sign of the last number in the Sturm sequence is the same as the sign of $-p\Delta$, where $\Delta = (n-1)^{n-1}p^n - n^n q^{n-1}$. The following table shows the succession of signs for the four possible combinations of values of p, Δ:

		f	f_1	f_2	f_3
1. $p > 0$, $\Delta > 0$	$-\infty$	+	−	+	−
	$+\infty$	+	+	−	−
2. $p < 0$, $\Delta > 0$	$-\infty$	+	−	−	+
	$+\infty$	+	+	+	+
3. $p > 0$, $\Delta < 0$	$-\infty$	+	−	+	+
	$+\infty$	+	+	−	+
4. $p < 0$, $\Delta < 0$	$-\infty$	+	−	−	−
	$+\infty$	+	+	+	−

Examination of this table shows that if $\Delta > 0$, the polynomial has two real roots; if $\Delta < 0$, the polynomial has no real root.

703 The Sturm sequence consists of the polynomials
$f = x^5 - 5ax^3 + 5a^2x + 2b$, $f_1 = x^4 - 3ax^2 + a^2$,
$f_2 = ax^3 - 2a^2x - b$, $f_3 = a(a^2x^2 - bx - a^3)$, $f_4 = a(a^5 - b^2)x$,
$f_5 = 1$.

If $\Delta = a^5 - b^2 > 0$, then $a > 0$, and every Sturm polynomial has positive leading coefficient. In this case all five roots of the polynomial f are real. If $\Delta < 0$, then the following table shows the possibilities depending on the sign of a.

		f	f_1	f_2	f_3	f_4	f_5
$a > 0$	$-\infty$	$-$	$+$	$-$	$+$	$+$	$+$
	$+\infty$	$+$	$+$	$+$	$+$	$-$	$+$
$a < 0$	$-\infty$	$-$	$+$	$+$	$-$	$-$	$+$
	$+\infty$	$+$	$+$	$-$	$-$	$+$	$+$

Thus the polynomial f has one real root if $\Delta < 0$.

704 Let f_λ, $f_{\lambda+1}$ be two consecutive polynomials of a "complete" Sturm sequence. If the leading coefficient in every polynomial has the same sign, then there are no sign changes for $x = +\infty$, and there is a sign change for $x = -\infty$, since the degrees are of different parity. If the highest coefficients have opposite sign, then for $x = +\infty$, f_λ, $f_{\lambda+1}$ exhibit a sign change but for $x = -\infty$, they do not. Thus if ν_1, ν_2 is the number of sign changes in the Sturm sequence for $-\infty$, $+\infty$, then $\nu_1 + \nu_2 = n$. On the other hand $\nu_1 - \nu_2$ is the number N of real roots of the polynomial. Therefore $\nu_2 = (n-N)/2$, which was to be shown.

705 The proof runs like the proof of Sturm's theorem except that the number of sign changes when x goes through a root of the original polynomial is not necessarily a unit.

706 The sequence of polynomials given in the problem is the Sturm series for the interval $x_0 \le x < +\infty$ and satisfies the conditions of problem 705 on the interval $-\infty < x \le x_0$. Therefore the number of roots of f in the interval (x_0, ∞) is $\nu(x_0) - \nu(+\infty)$; the number of roots of f in the interval $(-\infty, x_0)$ is $\nu(x_0) - \nu(-\infty)$, where ν is the number of changes of sign of the polynomial over the interval in question.

In general the number of real roots is equal to

$$2\nu(x_0) - \nu(+\infty) - \nu(-\infty).$$

707 We use Euler's theorem in the form

$$P_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n} = (-1)^n e^{\frac{x^2}{2}} \frac{d^{n-1}\left(-xe^{-\frac{x^2}{2}}\right)}{dx^{n-1}}$$

to obtain

$$P_n (-1)^{n-1} e^{\frac{x^2}{2}} \left(x \frac{d^{n-1} e^{-\frac{x^2}{2}}}{dx^{n-1}} + (n-1) \frac{d^{n-2} e^{-\frac{x^2}{2}}}{dx^{n-2}} \right)$$

Thus

$$P_n = xP_{n-1} = (n-1) P_{n-2}.$$

On the other hand we can differentiate the equation that defines P_{n-1}, and obtain

$$P'_{n-1} = (-1)^{n-1} x e^{\frac{x^2}{2}} \frac{d^{n-1} e^{-\frac{x^2}{2}}}{dx^{n-1}} + (-1)^{n-1} e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n},$$

so that

$$P'_{n-1} = xP_{n-1} - P.$$

Comparing this result with the previous formula, we obtain $P'_{n-1} = (n-1)P_{n-2}$ and therefore $P'_n = nP_{n-1}$.

These formulas show that the sequence $P_n, P_{n-1}, \ldots, P_1, P_0 = 1$ is a Sturm sequence for the polynomial P_n, since P_{n-1} is a positive constant multiplied by P'_n, and $P_{\lambda-1}$ is the negative of the remainder when $P_{\lambda+1}$ is divided by P_λ (except for a positive factor).

Every polynomial P_n has +1 for highest leading coefficient. Therefore all the roots of P_n are real.

708 By differentiating the equation defining P_n, we obtain

$$P'_n = (-1)^n e^x \frac{d^n (x^n e^{-x})}{dx^n} + (-1)^n e^x \frac{d^n (nx^{n-1}e^{-x} - x^n e^{-x})}{dx^n};$$

thus

$$P'_n = (-1)^n n e^x \frac{d^n x^{n-1} e^{-x}}{dx^n}.$$

From this we obtain

$$P_n = (-1)^n e^x \frac{d^n (x \cdot x^{n-1} e^{-x})}{dx^n}$$

$$= (-1)^n e^x \left[x \frac{d^n x^{n-1} e^{-x}}{dx^n} + n \frac{d^{n-1} x^{n-1} e^{-x}}{dx^{n-1}} \right] = \frac{x}{n} P'_n - nP_{n-1};$$

thus $xP'_n = nP_n + n^2 P_{n-1}$.

On the other hand,

$$P'_n = (-1)^n n e^x \frac{d^{n-1} [(n-1) x^{n-2} e^{-x} - x^{n-1} e^{-x}]}{dx^{n-1}},$$

thus $P'_n = -nP'_{n-1} + nP_{n-1}$. . Multiplying by x and

replacing xP_n', xP_{n-1}' by their values in terms of P_n, P_{n-1}, P_{n-2}, we obtain

$$P_n = (x - 2n + 1) P_{n-1} - (n-1)^2 P_{n-2}.$$

This relation shows that the polynomials P_n cannot be 0 simultaneously, and if $P_{n-1} = 0$, then P_n, P_{n-2} have opposite signs. Thus from

$$\frac{P_{n-1}}{P_n} = -\frac{1}{n} + \frac{xP_n'}{n^2 P_n}$$

we see that P_{n-1}/P_n changes sign from minus to plus when x goes through a root of P_n. Therefore the sequence P_n, $P_{n-1}, \cdots, P_1$, $P_0 = 1$ is a Sturm sequence for P_n on the interval $(0, \infty)$. The highest coefficient of every polynomial P_n is 1. $P_n(0) = (-1)^n n!$ Therefore $\nu(0) - \nu(+\infty) = n$, that is P_n has n positive roots.

709 From the relations $E_n' = E_{n-1}$, $E_n = E_{n-1} - (-x^n/n!)$, it follows that the polynomials E_n, E_{n-1} and $-x^n/n!$ form a Sturm series for E_n on the interval $-\infty$, $-\varepsilon$, if ε is a sufficiently small positive number. The sign changes on this interval are given by the following table:

$$\begin{array}{c|ccc} -\infty & (-1)^n & (-1)^{n-1} & (-1)^{n-1} \\ \hline -\varepsilon & + & + & (-1)^{n-1} \end{array}.$$

Therefore if n is even the polynomial E_n has no negative root; if n is odd the polynomial E_n has one negative root. Finally it is obvious that $E_n(x) > 0$ for $x \geq 0$.

710 The first step is to apply Euler's formula to the identity

$$\frac{d^{n+1}\left(x^2 e^{\frac{1}{x}}\right)}{dx^{n+1}} = \frac{d^n\left[(2x-1)\,e^{\frac{1}{x}}\right]}{dx^n}.$$

We obtain

$$x^2\,\frac{d^{n+1}e^{\frac{1}{x}}}{dx^{n+1}} + 2(n+1)\,x\,\frac{d^n e^{\frac{1}{x}}}{dx^n} + (n+1)\,n\,\frac{d^{n-1}e^{\frac{1}{x}}}{dx^{n-1}} =$$

$$= (2x-1)\,\frac{d^n e^{\frac{1}{x}}}{dx^n} + 2n\,\frac{d^{n-1}e^{\frac{1}{x}}}{dx^{n-1}},$$

Thus $P_n = (2nx+1)P_{n-1} - n(n-1)P_{n-2}x^2$. . On the other hand from the differential equation defining P_{n-1} we obtain

$$P_n = (2nx+1)P_{n-1} - x^2 P'_{n-1}.$$

This result shows that $P'_{n-1} = n(n-1)P_{n-2}$ and therefore $P'_n = (n+1)nP_{n-1}$. Putting these facts together we see that the series $P_n, P_{n-1}, P_{n-2}, \ldots, P_0 = 1$ is a Sturm series for P_n. Every polynomial P_n has positive leading coefficient. Therefore all the roots of P_n are real.

711 Two forms of the relation

$$\frac{d^n\left(\frac{x^2}{x^2+1}\right)}{dx^n} = -\,\frac{d^n\left(\frac{1}{x^2+1}\right)}{dx^n},$$

give

$$P_n - 2xP_{n-1} + (x^2+1)P_{n-2} = 0.$$

From the formula defining P_{n-1}, we obtain by differentiation $P_n = 2xP_{n-1} - \frac{x^2+1}{n}P'_{n-1}$;

thus $P'_{n-1} = nP_{n-2}$. Therefore $P'_n = (n+1)P_{n-1}$.

From the computations just performed, we see that $P_n, P_{n-1}, \ldots, P_0 = 1$ form a Sturm sequence for P_n. Every polynomial in this sequence has positive leading coefficient. Thus all roots of P_n are real.

The problem admits a direct solution. One has

$$\frac{1}{x^2+1} = \frac{1}{2i}\left(\frac{1}{x-i} + \frac{1}{x+i}\right).$$

From this one computes directly that

$$P_n(x) = \frac{1}{2i}\left[(x+i)^{n+1} - (x-i)^{n+1}\right].$$

It is easy to see that the roots of P_n are $\cot\{k\pi/(n+1)\}$, $k = 1, 2, \cdots, n$.

712 We apply Euler's formula to the identity

$$\frac{d^n \dfrac{x^2+1}{\sqrt{x^2+1}}}{dx^n} = \frac{d^{n-1} \dfrac{x}{\sqrt{x^2+1}}}{dx^{n-1}},$$

and obtain

$$P_n - (2n-1)xP_{n-1} + (n-1)^2(x^2+1)P_{n-2} = 0.$$

The equation defining P_{n-1} gives, on differentiation, the relation

$$P_n - (2n-1)xP_{n-1} + (x^2+1)P'_{n-1} = 0.$$

Thus

$$P'_{n-1} = (n-1)^2 P_{n-2} \quad , \quad P'_n = n^2 P_{n-1}.$$

From the relations already obtained, it follows that $P_n, P_{n-1}, \cdots, P_0 = 1$ form a Sturm sequence.

Each polynomial P_n has positive leading coefficient. Hence all roots of P_n are positive.

713 The function $F(x)$, $F'(x)$, $[f'(x)]^2$ form a Sturm sequence for F. The leading coefficients of this sequence: $3a_0^2$, $12a_0^2$, $9a_0^2$ are positive. Therefore the number of changes in sign for $x = -\infty$ differs from the number for $x = +\infty$ by 2.

If f has a double root, then F has one triple root and one simple one. If f has a triple root then F has a four-fold root.

714 If some polynomial of a Sturm sequence has a multiple root x_0 or a complex root α, then this polynomial can be replaced by a polynomial of lower degree obtained by dividing out the positive value $(x - x_0)^2$ or $(x - \alpha)(x - \bar{\alpha})$. The succeeding polynomials are obtained in the usual way by applying the Euclidean algorithm and reversing the sign of the remainder. When this is done the number of changes of sign for $x = -\infty$ becomes $\leq n - 2$, where n is the degree of the original polynomial. Therefore the number of real roots must be $\leq n - 2$.

715 Let $F(x) = (x^2 - 1)^n$. $F(x)$ has $-1, +1$ as n-fold roots. $F'(x)$ has $-1, +1$ as $n - 1$-fold roots. Moreover by Rolle's theorem there is an additional root in the interval $-1, +1$. $F''(x)$ has $-1, +1$ as $n - 2$-fold roots, and two additional roots in the open interval $-1, +1$; etc. $F^{(n)}(x) = P_n(x)$ has n roots in the open interval $-1, 1$.

716 Let $x_1, \cdots, x_k$ be the distinct roots of $f(x)$, $x_1 < x_2 < x_3 < \cdots < x_k$; let their multiplicities be respectively $\alpha_1, \alpha_2, \cdots, \alpha_k$. The function $\varphi(x) = f'(x)/f(x)$ is continuous in the open intervals $(-\infty, x_1)$, $(x_1, x_2), \cdots, (x_{k-1}, x_k)$, $(x_k, +\infty)$ and changes from 0 to $-\infty$ in each of the intervals (x_{i-1}, x_i) and from $+\infty$ to 0 in the interval (x_k, ∞). This is clear because the roots are distinct and $\varphi(x)$ changes sign at x_i.

Therefore $\varphi(x) + \lambda$ has a root in each of the intervals (x_{i-1}, x_i) and moreover for $\lambda > 0$ one root in the interval $(-\infty, x_1)$ and for $\lambda < 0$ one root in the interval $(x_i, +\infty)$.

Thus, $\varphi(x) + \lambda$ and therefore $f(x)[\varphi(x)+\lambda] = \lambda f(x) + f'(x)$ has k roots distinct from $x_1, x_2, \cdots, x_k$, for $\lambda \neq 0$, or k - 1 roots distinct from $x_1, x_2, \cdots, x_k$ for $\lambda = 0$. Moreover $\lambda f(x) + f'(x)$ has $x_1, x_2, \cdots, x_k$ as roots of multiplicities $\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_k - 1$. respectively. This establishes the general proposition that the number of real roots of the polynomial $\lambda f(x) + f'(x)$, counted according to multiplicity, is $\alpha_1 + \alpha_2 + \ldots + \alpha_k$ if $\lambda \neq 0$; and is $\alpha_1 + \alpha_2 + \ldots + \alpha_k - 1$ if $\lambda = 0$. Thus the number of such roots is the same as the degree of the polynomial $\lambda f(x) + f'(x)$.

717 Let

$g(x) = a_0(x+\lambda_1)(x+\lambda_2)\ \dots\ (x+\lambda_n)$,

$F_0(x) = a_0 f(x)$,

$F_1(x) = F_0(x) + \lambda_1 F_0'(x) = a_0 f(x) + a_0\lambda_1 f'(x)$,

$F_2(x) = F_1(x) + \lambda_2 F_1'(x) = a_0 f(x) + a_0(\lambda_1+\lambda_2) f'(x) + a_0\lambda_1\lambda_2 f''(x)$

etc.

Then $F_n(x) = F_{n-1}(x) + \lambda_n F_{n-1}'(x) = a_0 f(x) + a_1 f'(x) + \dots + a_n f^{(n)}(x)$, where $a_0, a_1, \cdots, a_n$ are the coefficients of g. By the result of problem 715, all the roots of all the polynomials $F_0, F_1, \cdots, F_n$ are real.

718 The polynomial $a_0x^n + a_1mx^{n-1} + \dots + m(m-1)\dots(m-n+1)a_n$ $= [a_0x^m + a_1(x^m)' + \dots + a_n(x^m)^{(n)}]\, x^{n-m}$, and all the roots of x^m are real.

719 The polynomial $a_nx^n + na_{n-1}x^{n-1} + n(n-1)a_{n-2}x^{n-2} + \dots + a_0 n!$ has all its roots real. Therefore all the roots of $a_0 n!x^n + a_1 n(n-1)\dots 2x^{n-1} + \dots + na_{n-1}x + a_n$ are real. Using the result of problem 718 again we see that all the roots of the polynomial
$a_0 n!x^n + a_1 n\cdot n(n-1)\dots 2x^{n-1} + a_2 n(n-1)\cdot n(n-1)\dots 3x^{n-2}$
$+ \dots + a_n n!$
are real. It only remains to divide by n!

720 All the roots of the polynomial

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \dots + x^n$$

are real. The problem is completed by referring to 719.

721 The polynomial $f(x) = nx^n - x^{n-1} - x^{n-2} - \dots - 1$ has 1 as a root. The trick is to consider $F(x) = (x-1)f(x) = nx^{n+1} - (n+1)x^n + 1$. Then $F'(x) = n(n+1)(x-1)x^{n-1}$. If n is odd the

polynomial $F(x)$ has a single extremum, namely a minimum at $x = 1$. Therefore it has no other real roots except for the double root $x = 1$. If n is even, $F(x)$ increases monotonically from $-\infty$ to 1 for $-\infty < x \leq 0$, decreases from 1 to 0 for $0 \leq x \leq 1$ and increases from 0 to ∞ for $1 \leq x < \infty$. Therefore $F(x)$ has an additional root in this case.

722 The derivative of the polynomial in this problem is positive for all real values of x. Thus the polynomial has a single real root.

723 Let $a < b < c$; $f(-\infty) < 0$; $f(a) = B^2(b - a) + C^2(c - a) > 0$; $f(c) = - A^2(c - a) - B^2(c - b) < 0$; $f(+\infty) > 0$. Therefore f has real roots in the intervals $(-\infty, a)$; (a, c); $(c, +\infty)$.

724 Suppose

$$a_1 < a_2 < \cdots < a_n.$$

Set

$$\varphi(a+bi)=B+\sum_{k=1}^{n}\frac{A_k^2}{a+bi-a_k}=B+\sum_{k=1}^{n}\frac{A_k^2(a-a_k-bi)}{(a-a_k)^2+b^2};$$

$$\operatorname{Im}(\varphi(a+bi))=-b\sum_{k=1}^{n}\frac{A_k^2}{(a-a_k)^2+b^2}\neq 0$$

for $b \neq 0$, and every term in the summation sign is positive. Therefore $\varphi(a + bi) \neq 0$ for $b \neq 0$. Another way to obtain this result is to note that $\varphi(x)$ decreases from $+\infty$ to $-\infty$ when x increases from a_i to a_{i+1}; $\varphi(x)$ decreases from 0 to $-\infty$ on $(-\infty, a_1)$; $\varphi(x)$ decreases from $+\infty$ to 0 on (a_n, ∞).

725 $\frac{f'(x)}{f(x)} \sum_{k=1}^{n} \frac{1}{x - x_k}$, where x_k are the roots of f(x).

Then

$$[f'(x)]^2 - f(x)f''(x) = [f(x)]^2 \sum_{k=1}^{n} \frac{1}{(x-x_k)^2} > 0$$

for every real x.

726 Let $x_1 < x_2 < \dots < x_n$ be the roots of the polynomial f(x), and $y_1 < y_2 < \dots < y_m$ be roots of the polynomial $\varphi(x)$.

The hypothesis shows that either m = n, m = n - 1, or m = n + 1. A simple change of notation brings us to one of the following two cases:

$$x_1 < y_1 < x_2 < y_2 < \dots < y_{n-1} < x_n$$

or

$$x_1 < y_1 < x_2 < y_2 < \dots < y_{n-1} < x_n < y_n.$$

Suppose $\lambda \neq 0$. We write the equation to be investigated

$$f(x)/\varphi(x) = -\mu/\lambda,$$

and set $\psi(x) = f(x)/\varphi(x)$.

If m = n, then $\psi(x)$ decreases: from a_0/b_0 to $-\infty$ for $-\infty < x < y_1$, taking the value 0 at $x = x_1$;

from $+\infty$ to $-\infty$ for $y_k < x < y_{k+1}$, taking the value 0 at $x = x_{k+1}$;

from $+\infty$ to a_0/b_0 for $y_n < x < +\infty$.

Here we have to know that the leading coefficients of f(x), $\varphi(x)$ are a_0, b_0 and assume them positive.

Now the function $\psi(x)$ is continuous in each of the above intervals. Thus the equation $\psi(x) = -\mu/\lambda$ has n real roots if $-\mu/\lambda \neq a_0/b_0$, and has n - 1 real roots in the contrary case. Thus the number of real roots of the equation $\lambda f(x) + \mu \varphi(x)$ is equal to its degree.

When m = n - 1, the argument is similar.

727 Since the polynomial F(x) reduces respectively to f(x), $\varphi(x)$ in the cases $\lambda = 1$, $\mu = 0$; $\mu = 1$, $\lambda = 0$, all roots of f(x), $\varphi(x)$ are real.

Assume that the roots of f(x), $\varphi(x)$ do not interlace. The only case to consider is that in which the polynomial f(x) has two successive roots x_1, x_2 and the polynomial $\varphi(x)$ has no roots in the interval (x_1, x_2). Since the quotient $\psi(x) = f(x)/\varphi(x)$ is continuous on the interval (x_1, x_2) in this case and reduces to zero at the end points, Rolle's theorem shows that the derivative $\psi'(x_0)$ is 0 at an intermediate point x_0 of this interval. Thus the polynomial $\psi(x) - \psi(x_0)$ has x_0 as a k-fold root, $k \geq 2$. The fundamental result of problem 581 shows that there is a small circle $|z - x_0| = \rho$ on the circumference of which there are at least four points with $\operatorname{Im}(\psi(z)) = \operatorname{Im}(\psi(x_0)) = 0$.

Of these four points at least one, z_0, is not real. The number $\mu = \psi(z_0)$ is real. (Just established.) The polynomial $F(x) = -f(x) + \mu\varphi(x)$

has a nonreal root, but this contradicts the hypothesis.

728 Let $\xi_1 < \xi_2 < \dots < \xi_{n-1}$ be the roots of the polynomial f'(x). These numbers divide the real axis into n intervals:

$$(-\infty, \xi_1), (\xi_1, \xi_2), \dots, (\xi_{n-2}, \xi_{n-1}), (\xi_{n-1}, \infty).$$

By Rolle's theorem the polynomial f(x) has no more than one root in each of these intervals. Further for arbitrary λ, the polynomial $f'(x) + \lambda f''(x)$ has no more than one root in each of the intervals in question. Therefore the polynomial $f(x) + \lambda f'(x)$ has no more than two roots in each of the intervals, multiplicity being counted.

Divide the intervals into two categories. The first category consists of the intervals containing roots of f(x). The second category consists of the intervals containing no roots of f(x). Consider the function $\psi(x) = f(x)/f'(x)$. In every interval of the first category $\psi(x)$ has a single root and therefore changes sign on the interval. In every interval of the second category $\psi(x)$ has an invariable sign. Thus in every interval of the first category, $\psi(x)+\lambda$ has an odd number of roots, multiplicity being counted. The only odd number no greater than 2 is 1. Therefore $\psi(x)+\lambda$ has a single root in each interval of the first category. Therefore

$\psi'(x)$ has no root in an interval of the first category.

In an interval of the second category we reason as follows. Suppose the absolute value of $\psi(x)$ assumes a minimum at the point ξ_0 interior to an interval of the second category: $\lambda_0 = \psi(\xi_0)$. Suppose for definiteness that $\psi(x)$ is positive in this interval. Then the function $\psi(x) - \lambda$ has no root in the interval if $\lambda < \lambda_0$; and at least two roots if $\lambda > \lambda_0$.

The arguments in the first paragraph show that for $\lambda > \lambda_0$, $\psi(x) - \lambda$ has exactly two roots in an interval of the second category and both are simple. Thus $\psi(x) - \lambda_0$ has ξ_0 as a multiple root of multiplicity two.

Summarizing, we see that $\psi(x) - \lambda$ has no multiple roots in an interval of the second category for just one value of λ. Clearly every root of η $f'^2(x) - f(x)\, f''(x)$ is a multiple root of $\psi(x) - \psi(\eta)$, since

$$[\psi(x)]' = \frac{f'^2(x) - f(x) f''(x)}{[f'(x)]^2}.$$

Thus the number of real roots of $[f'(x)]^2 - f(x)\, f''(x)$ is the same as the number of intervals in the second category. This is clearly the number of imaginary roots of $f(x)$.

729 Problem 727 shows that the roots of $f_1'(x)$, $f_2'(x)$ interlace if all the roots of $\lambda f_1'(x)+\mu f_2'(x)$ are real. This follows by applying Rolle's theorem to the combination $\lambda f_1(x)+\mu f_2(x)$ since all the roots of the latter are real for arbitrary λ, μ. (See problem 726.)

730 Let f(x) have no multiple roots, and suppose $\xi_1 < \xi_2 < \cdots < \xi_{n-1}$ are the roots of f'(x). Consider the function $\psi(x)=\frac{f(x)}{f'(x)}+\frac{x+\lambda}{\gamma}$. Obviously $\lim_{x\to\infty}\frac{\psi(x)}{x}=\frac{1}{n}+\frac{1}{\gamma}>0$, for $\gamma>0$, and for $\gamma<-n$. Thus $\psi(x)\to+\infty$ for $x\to+\infty$, and $\psi(x)\to-\infty$ for $x\to-\infty$. Moreover, $\psi(x)\to-\infty$ for $x\to\xi_i+$; $\psi(x)\to+\infty$ for $x\to\xi_i-$. Thus $\psi(x)$ increases from $-\infty$ to $+\infty$ in each of the intervals $(-\infty,\ \xi_1)$, $(\xi_1,\ \xi_2)$, $\ldots$, $(\xi_{n-1},\ \infty)$, and is continuous in the interiors of these intervals.

Thus $\psi(x)$, and therefore its numerator $\gamma f(x)+(x+\lambda)f'(x)$ have no fewer than n distinct roots for $\gamma>0$, or for $\gamma<-n$. Now $\gamma f(x)+(x+\lambda)f'(x)$ is a polynomial of degree n; thus it has at most n roots. If f(x) has a multiple root and $x_1, x_2, \cdots, x_k$ are the distinct roots of f(x), then f'(x) has k - 1 roots $\xi_1, \xi_2, \ldots, \xi_{k-1}$, distinct from $x_1, x_2, \cdots, x_k$. This is a general fact that follows from the study of the graph of the polynomial. Now we simply

factor off the necessary factors $(x - x_1)^{j}, \cdots,$ from $\psi(x)$ and use the arguments above for the complementary factors. To do this we have only to know that $\xi_1, \xi_2, \cdots, \xi_{k-1}$ are not roots of $f(x)$ unless $-\lambda$ is a root of $f(x)$. If $-\lambda$ is not a root of $f(x)$, the number of real roots of $\gamma f(x)+(x+\lambda)f'(x)$ is $n - k$ (to allow for the multiple roots) plus k, to allow for the remaining roots. If $-\lambda$ is a root of $f(x)$, the first summand becomes $n - k + 1$, the second summand $k - 1$.

Thus the polynomial $\gamma f(x)+(\lambda+x)f'(x)$ has n roots in all cases, multiplicity being counted.

731 Set $\varphi(x)=b_k(x+\gamma_1)(x+\gamma_2)\ldots(x+\gamma_k)$. Each of the numbers γ_i is either positive or less than $-n$: $\gamma_i > 0$ or $\gamma_i < -n$.

Obviously the coefficients of the polynomial

$$F_1(x)=\gamma_1 f(x)+xf'(x) \text{ are } a_i(\gamma_1+i),$$

and the coefficients of the polynomial

$$F_2(x)=\gamma_2 F_1(x)+xF_1'(x) \text{ are } a_i(\gamma_1+i)(\gamma_2+i);$$

etc. The coefficients of the polynomial

$$F_k(x)=\gamma_k F_{k-1}(x)+xF_{k-1}'(x)$$

are

$$a_i(\gamma_1+i)(\gamma_2+i)\ldots(\gamma_k+i),\ i=1,\ 2,\ \ldots,\ n.$$

The fundamental result of problem 730 shows that all roots of all the polynomials $F_1, F_2, \cdots, F_k$

are real. But

$$a_0\varphi(0) + a_1\varphi(1)x + \ldots + a_n\varphi(n)x^n = b_k F_k(x).$$

732 Set $f(x) = f_1(x)(x + \lambda)$, where λ is real and $f_1(x)$ is a polynomial of degree $(n - 1)$, having real roots. As an induction hypothesis suppose the theorem established for polynomials of degree $(n - 1)$.

Set

$$f_1(x) = b_0 + b_1 x + \ldots + b_{n-1}x^{n-1},$$
$$f(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

Then

$$\begin{aligned} a_0 &= \lambda b_0, \\ a_1 &= \lambda b_1 + b_0, \\ a_2 &= \lambda b_2 + b_1, \\ &\ldots\ldots\ldots \\ a_{n-1} &= \lambda b_{n-1} + b_{n-2}, \\ a_n &= \qquad b_{n-1} \end{aligned}$$

and

$$\begin{aligned} &a_0 + a_1\gamma x + a_2\gamma(\gamma - 1)x^2 + \ldots + a_n\gamma(\gamma - 1)\ldots(\gamma - n + 1)x^n \\ &= \lambda[b_0 + b_1\gamma x + b_2\gamma(\gamma - 1)x^2 + \ldots + b_{n-1}\gamma(\gamma - 1)\ldots \\ &\ldots(\gamma - n + 2)x^{n-1}] + x\gamma(b_0 + b_1(\gamma - 1)x + b_2(\gamma - 1)(\gamma - 2)x^2 + \ldots \\ &\ldots + b_{n-1}(\gamma - 1)(\gamma - 2)\ldots(\gamma - n + 1)x^{n-1}) \\ &= \lambda\varphi(x) + x[\gamma\varphi(x) - x\varphi'(x)], \end{aligned}$$

where $\varphi(x)$ denotes the polynomial

$$b_0 + b_1\gamma x + b_2\gamma(\gamma - 1)x^2 + \ldots + b_{n-1}\gamma(\gamma - 1)\ldots(\gamma - n + 2)x^{n-1}.$$

The induction hypothesis allows us to assume that all roots of the polynomial $\varphi(x)$ are real. To complete the argument we shall establish the following lemma.

Lemma. If $\varphi(x)$ is a polynomial of degree $n-1$ having real roots, then all roots of the polynomial $\psi(x)=\lambda\varphi+\gamma x\varphi-x^2\varphi'$ are real for $\gamma>n-1$, for arbitrary real λ.

Proof. If 0 is a root of φ, we can factor it off as follows: $\varphi=x^k\varphi_1$, $\varphi_1(0)\neq 0$,

$$\psi(x)=x^k(\lambda\varphi_1+(\gamma-k)x\varphi_1-x^2\varphi_1')=x^k\psi_1,$$

and $\gamma_1=\gamma-k$ exceeds the degree of φ_1.

Now suppose that 0 is not a root of φ, and let $x_1, x_2, \cdots, x_m$ be its distinct roots. The polynomial ψ will have these same numbers as roots, with multiplicity sum $n-1-m$. Now consider

$$w(x)=\lambda+\gamma x-\frac{x^2\varphi'(x)}{\varphi(x)}.$$

It is obvious that

$$\lim_{x\to\infty}\frac{w(x)}{x}=\gamma-(n-1)>0.$$

Therefore $w(x)\to-\infty$ for $x\to-\infty$ and $w(x)\to+\infty$ for $x\to+\infty$. Moreover $w(x)\to+\infty$ for $x\to x_i+$, $w(x)\to-\infty$ for $x\to x_i-$. Thus $w(x)$ has roots in each of the intervals

$$(-\infty, x_1), (x_1, x_2), \ldots, (x_{m-1}, x_m), (x_m, +\infty).$$

It follows that the number of real roots of $\psi(x)$, multiplicity being counted is $n-1-m+m+1=n$. This is the degree of $\psi(x)$, as was to be shown.

733 If all roots of the polynomial $a_0+a_1x+\ldots+a_nx^n$ are real, then all roots of the polynomial $a_0x^n+a_1x^{n-1}+\ldots+a_n$ are also real. Now all roots of the polynomials

$$a_0\gamma_1(\gamma_1-1)\ldots(\gamma_1-n+1)x^n+a_1\gamma_1(\gamma_1-1)\ldots$$
$$\ldots(\gamma_1-n+2)x^{n-1}+\ldots+a_{n-1}\gamma_1x+a_n$$

and

$$a_0\gamma_1(\gamma_1-1)\ldots(\gamma_1-n+1)+a_1\gamma_1(\gamma_1-1)\ldots(\gamma_1-n+2)x+\ldots$$
$$\ldots+a_{n-1}\gamma_1x^{n-1}+a_nx^n=\Big[a_0+\frac{a_1}{\gamma_1-n+1}x+\ldots$$
$$+\frac{a_{n-1}}{(\gamma_1-n+1)(\gamma_1-n+2)\ldots(\gamma_1-1)}x^{n-1}$$
$$+\frac{a_n}{(\gamma_1-n+1)(\gamma_1-n+2)\ldots\gamma_1}x^n\Big]\gamma_1(\gamma_1+1)\ldots(\gamma_1-n+1)$$

are real for $\gamma_1>n-1$. Set $\gamma_1-n+1=\alpha>0$, and note that all roots of the polynomial

$$a_0+\frac{a_1}{\alpha}x+\frac{a_2}{\alpha(\alpha+1)}x^2+\ldots+\frac{a_n}{\alpha(\alpha+1)\ldots(\alpha+n-1)}x^n$$

are real. The result of the exercise follows from a second application of problem 732.

734 1. Suppose that all roots of f(x) are positive. Then the polynomial $a_0+a_1wx+\ldots+a_nw^{n^2}x^n$ has no negative roots.

Suppose the theorem established for polynomials of degree n - 1. Suppose

$$\varphi(x)=b_0+b_1wx+b_2w^4x^2+\ldots+b_{n-1}w^{(n-1)^2}x^{n-1}.$$

Set $0<x_1<x_2<\ldots<x_{n-1}$, where $x_1, x_2, \ldots, x_{n-1}$ are the roots of $\varphi(x)$ and suppose $\frac{x_i}{x_{i-1}}>w^{-2}$.

Now set $f(x)=(\lambda-x)(b_0+b_1x+\ldots+b_{n-1}x^{n-1})$.
The coefficients of the polynomial f(x) are

$$\begin{aligned}
a_0 &= \lambda b_0,\\
a_1 &= \lambda b_1 - b_0,\\
a_2 &= \lambda b_2 - b_1,\\
&\ldots\ldots\ldots\\
a_{n-1} &= \lambda b_{n-1} - b_{n-2},\\
a_n &= -b_{n-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\psi(x) &= a_0+a_1wx+a_2w^4x^2+\ldots+a_nw^{n^2}x^n=\lambda(b_0+b_1wx+\ldots\\
&\ldots+b_{n-1}w^{(n-1)^2}x^{n-1})-x(b_0w+b_1w^4x+\ldots+b_{n-1}w^{n^2}x^{n-1})\\
&=\lambda\varphi(x)-xw\varphi(xw^2).
\end{aligned}$$

The roots of the polynomial $\varphi(x)$, $x\varphi(xw^2)$ interlace by the induction hypothesis. Therefore all the roots of the polynomial $\lambda\varphi(x) + xw\varphi(xw^2)$ are real. It remains to check that they are distributed in the same way as the roots of the polynomial $\varphi(x)$.

Suppose the roots of $\psi(x)$ are $z_1, z_2, \cdots, z_n$. It is easy to see that

$$0<z_1<x_1<x_1w^{-2}<z_2<x_2<x_2w^{-2}<z_3$$
$$\ldots<z_{n-1}<x_{n-1}<x_{n-1}w^{-2}<z_n.$$

Furthermore, $\frac{z_i}{z_{i-1}}>\frac{x_{i-1}w^{-2}}{x_{i-1}}=w^{-2}$, and the assertion is established.

2. Consider

$$\varphi_m(x)=\left(1-\frac{x^2\lg\frac{1}{w}}{m}\right)^m.$$

If m is sufficiently large, the roots of the polynomial $\varphi_m(x)$ are $\pm\sqrt{\frac{m}{\lg\frac{1}{w}}}$; these do

not lie in the interval (0, n). Thus, by problem 731, all the roots of the polynomial $a_0\varphi_m(0)+a_1\varphi_m(1)x+\ldots+a_n\varphi_m(n)x^n$ are real. But $\lim_{m\to\infty}\varphi_m(x)=w^{x^2}$. Thus all the roots of $a_0+a_1wx+\ldots+a_nw^{n^2}x^n$ are real, since the roots of a polynomial are continuous functions of the coefficients.

735 Let the roots of the polynomial $f(x)=a_0+a_1x+\ldots+a_nx^n$ be $x_1, x_2, \cdots, x_n$. Without loss of generality we may assume these to be positive. Now set

$$\varphi(x)=a_0\cos\varphi+a_1\cos(\varphi+\theta)x+\ldots+a_n\cos(\varphi+n\theta)x^n,$$
$$\psi(x)=a_0\sin\varphi+a_1\sin(\varphi+\theta)x+\ldots+a_n\sin(\varphi+n\theta)x^n.$$

Then

$$\varphi(x)+i\psi(x)=(\cos\varphi+i\sin\varphi)a_n\prod_{i=1}^{n}(\alpha x-x_i),$$

$$\varphi(x)-i\psi(x)=(\cos\varphi-i\sin\varphi)a_n\prod_{i=1}^{n}(\alpha' x-x_i),$$

where

$$\alpha=\cos\theta+i\sin\theta,\ \alpha'=\cos\theta-i\sin\theta.$$

Thus

$$\Phi(x)=\frac{\varphi(x)+i\psi(x)}{\varphi(x)-i\psi(x)}=\frac{\cos\varphi+i\sin\varphi}{\cos\varphi-i\sin\varphi}\prod_{i=1}^{n}\frac{\alpha x-x_i}{\alpha' x-x_i}.$$

Let $x = \rho\beta$ be a root of the polynomial $\varphi(x)$; $\rho = |x|$; $\beta = \cos\lambda + i\sin\lambda$. Then $|\Phi(x)|=1$, and therefore

$$\prod_{i=1}^{n}\left|\frac{\rho\alpha\beta-x_i}{\rho\alpha'\beta-x_i}\right|=1,$$

But

$$\left|\frac{\rho\alpha\beta-x_i}{\rho\alpha'\beta-x_i}\right|^2=\frac{(\rho\alpha\beta-x_i)\,(\rho\alpha'\beta'-x_i)}{(\rho\alpha'\beta-x_i)\,(\rho\alpha\beta'-x_i)}=$$

$$=1+\frac{\rho x_i(\alpha-\alpha')(\beta'-\beta)}{|\rho\alpha'\beta-x_i|^2}=1+\frac{4\rho x_i\sin\theta\sin\lambda}{|\rho\alpha'\beta-x_i|^2}.$$

The case $\sin\theta = 0$ is not interesting.

If $\sin\lambda \neq 0$, then all the expressions $\left|\frac{\rho\alpha\beta-x_i}{\rho\alpha'\beta-x_i}\right|^2$ are simultaneously greater than unity or simultaneously less than unity and their product cannot be 1. Therefore we must have $\sin\lambda = 0$, that is, x real.

736 Let $x_1, x_2, \cdots, x_n$ be roots of the polynomial $f(x)=a_0+ib_0+(a_1+ib_1)x+\ldots+(a_n+ib_n)x^n=\varphi(x)+i\psi(x)$. Each of these roots has positive imaginary part. Consider the polynomial $\bar{f}(x)=\varphi(x)-i\psi(x)$. Clearly this polynomial has roots $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$. Thus

$$\Phi(x)=\frac{\varphi(x)+i\psi(x)}{\varphi(x)-i\psi(x)}=\prod_{i=1}^{n}\frac{x-x_i}{x-x_i'}\cdot\frac{a_n+ib_n}{a_n-ib_n}.$$

For every root x_0 of the polynomial $\varphi(x)$ the relation

$$|\Phi(x_0)|=\prod_{i=1}^{n}\left|\frac{x_0-x_i}{x_0-x_i'}\right|=1 \text{ holds.}$$

But

$$\left|\frac{x_0-x_l}{x_0-x_l'}\right|^2=\frac{(x_0-x_l)(x_0'-x_l')}{(x_0-x_l')(x_0'-x_l)}=1+\frac{(x_l-x_l')(x_0-x_0')}{|x_0-x_l|^2}$$
$$=1-\frac{4\,\mathrm{Im}\,(x_0)\,\mathrm{Im}\,(x_l)}{|x_0-x_l'|^2}\,.$$

A simple geometric argument shows that if $\mathrm{Im}(x_0) > 0$, then $\left[\frac{x_0 - x_i}{x_0 - \bar{x}_i}\right] < 1$; in the contrary case this fraction is always greater than 1. Therefore the equality $|\Phi(x_0)|=1$ can be valid only if x_0 is real. That is, all roots of $\varphi(x)$ are real.

Now consider the polynomial

$$(\alpha-\beta i)[\varphi(x)+i\psi(x)]=\alpha\varphi(x)+\beta\psi(x)+i[\alpha\psi(x)-\beta\varphi(x)].$$

Its roots are the same as the roots of the polynomial under consideration. Therefore the real part, $\alpha\varphi(x)+\beta\psi(x)$ has real roots for arbitrary real α, β (not both zero). By problem 727 the roots of $\varphi(x)$, $\psi(x)$ interlace.

737 Let $\varphi(x)$ have roots $x_1, x_2, \cdots, x_n$; and let $\psi(x)$ have roots $y_1, y_2, \cdots, y_n$. It can be assumed without loss of generality that the leading coefficients are positive, and that

$$x_1 > y_1 > x_2 > y_2 > \ldots > y_{n-1} > x_n > y_n$$

where the last member may be absent.

We expand $\psi(x)/\varphi(x)$ into partial fractions:

$$\frac{\psi(x)}{\varphi(x)}=A+\sum_{k=1}^{n}\frac{A_k}{x-x_k}\,;\quad A_k=\frac{\psi(x_k)}{\varphi'(x_k)}.$$

It is easy to see that every numerator $A_k > 0$. Take $x = a + bi$ and compute the imaginary part of

$$\frac{-i(\varphi(x)+i\psi(x))}{\varphi(x)} = \frac{\psi(x)}{\varphi(x)} - i;$$

$$\operatorname{Im}\left(\frac{\psi(x)}{\varphi(x)} - i\right) = -1 + \operatorname{Im}\left(\sum_{k=1}^{n} \frac{A_k}{a+bi-x_k}\right) = -1 - b\sum_{k=1}^{n} \frac{A_k}{(a-x_k)^2+b^2}.$$

If $b \geq 0$, then $\operatorname{Im}\left(\frac{\psi(x)}{\varphi(x)} - i\right) < 0$; therefore

$\varphi(x) + i\psi(x) \neq 0$. Thus all roots of the polynomial $\varphi(x) + i\psi(x)$ would lie in the lower half plane. Similar arguments take care of the remaining cases.

738

$$\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{x-x_k}; \quad \text{where } x_k \text{ are the roots of } f(x).$$

Set $x = a - bi$, $b > 0$. Then

$$\operatorname{Im}\left(\frac{f'(a-bi)}{f(a-bi)}\right) = \sum_{k=1}^{n} \frac{b+\operatorname{Im}(x_k)}{(x-x_k)^2} > 0.$$

Thus

$$f'(a - bi) \neq 0.$$

739 Let the given half plane be defined by the inequality

$$r\cos(\theta - \varphi) > p, \text{ where } x = r(\cos\varphi + i\sin\varphi).$$

Set $x = (\overline{x} + pi)(\sin\theta - i\cos\theta)$. Then

$$\overline{x} = -pi + x(\sin\theta + i\cos\theta = r\sin(\theta - \varphi) + i[r\cos(\theta - \varphi) - p].$$

It is clear that if x lies in the given half plane, then $\overline{x}$ lies in the half plane $\operatorname{Im}(\overline{x}) > 0$,

and conversely. The roots of the polynomials $f[(\overline{x} + pi)(\sin\theta - i\cos\theta)]$, must therefore lie in the upper half plane. Problem 738 shows that the roots of its derivative, namely the roots of $[\sin\theta - i\cos\theta]\, f'[(\overline{x} + pi)(\sin\theta - i\cos\theta)]$, must also lie in the upper half plane.

Therefore the roots of the polynomial $f'(x)$ lie in the given half plane.

740 Follows directly from problem 739.

741 The roots of the given polynomial satisfy one of the equations:

$$\frac{f'(x)}{f(x)} + \frac{1}{ki} = 0 \quad , \quad \frac{f'(x)}{f(x)} - \frac{1}{ki} = 0.$$

If we separate these into partial fractions we obtain the two equations $\sum_{k=1}^{n} \frac{1}{x - x_k} \pm \frac{1}{ki} = 0,$

where x_k are the roots of $f(x)$, which are real by hypothesis. Set $x = a + bi$. Then

$$\left| \operatorname{Im}\left(\sum_{k=1}^{n} \frac{1}{x - x_k} \right) \right| = |b| \sum_{k=1}^{n} \frac{1}{(a - x_k)^2 + b^2} < \frac{n}{|b|}.$$

Thus every root of the given polynomial must satisfy $1/k < n/|b|$. Thus $|b| < kn$.

742 Denote the roots of $f'(x)$ by $\xi_1, \xi_2, \cdots, \xi_{n-1}$; these are obviously real. Let the roots of $f(x) - b$, $f(x) - a$ be $y_1, y_2, \cdots, y_n$; $x_1, x_2, \cdots, x_n$ respectively. Then

$$y_1 < \xi_1 < y_2 < \xi_2 < \ldots < y_{n-1} < \xi_{n-1} < y_n,$$
$$x_1 < \xi_1 < x_2 < \xi_2 < \ldots < x_{n-1} < \xi_{n-1} < x_n.$$

It follows from these equations that the intervals bounded by the points x_i, y_i do not overlap. Thus they are all included in one or another of the non-overlapping intervals

$$(-\infty, \xi_1); \quad (\xi_1, \xi_2); \ldots; (\xi_{n-1}, +\infty).$$

In each interval x_i, y_i the polynomial $f(x)$ starts at the value a and changes continuously to the value b. Therefore the polynomial $f(x) - \lambda$ is 0 for n distinct real values. The assertion is established if all roots are distinct. A very simple modification of the argument covers the case of multiple roots.

743 If the real parts of the roots of the polynomial $f(x) = x^n + a_1x^{n-1} + \ldots + a_n$ have the same sign, the imaginary parts of the roots of the polynomial

$$i^n f(-ix) = x^n + ia_1x^{n-1} - a_2x^{n-2} - ia_3x^{n-3} + \ldots$$

also have the same sign. The converse is true.

By problems 736, 737 necessary and sufficient conditions for this are that the roots of the polynomials $x^n - a_2x^{n-2} + a_4x^{n-4} - \ldots$, $a_1x^{n-1} - a_3x^{n-3} + a_5x^{n-5} - \ldots$ be real and interlace.

744 It must be true that $a > 0$ and that the roots of the polynomials $x^3 - bx$, $ax^2 - c$ be real and interlace. Necessary and sufficient conditions for this are $0 < c/a < b$, or $c > 0$, $ab - c > 0$.

Thus the conditions $a > 0$, $c > 0$, $ab - c > 0$ are necessary and sufficient conditions that the real parts of the roots of the given equation be negative.

745 $a > 0$, $c > 0$, $d > 0$, $abc - c^2 - a^2d > 0$.

746 Set $x = (1 + y)/(1 - y)$. It is easy to see that if $|x| < 1$, then the real part of y is negative and conversely.

Thus all three roots, x_1, x_2, x_3 of the equation $f(x) = 0$ will have absolute value less than 1, if and only if all roots of the equation $f(\{1 + y\}/\{1 - y\}) = 0$ have negative real part. This equation has the form

$$y^3(1-a+b-c)+y^2(3-a-b+3c)+y(3+a-b-3c) + (1+a+b+c)=0.$$

It is easy to see moreover that the condition

$$1-a+b-c=(1+x_1)(1+x_2)(1+x_3)>0$$

is necessary. Using the result of problem 744 we obtain the following necessary and sufficient conditions

$$1-a+b-c>0;\quad 1+a+b+c>0;\quad 3-a-b+3c>0;\quad 1-b+ac-c^2>0.$$

747 $$f(x)(1-x)=a_n+(a_{n-1}-a_n)x+ +(a_{n-2}-a_{n-1})x^2+\dots+(a_0-a_1)x^n-a_0x^{n+1}.$$

Set $|x| = \rho > 1$. Then

$$|f(x)(1-x)| \geqslant a_0\rho^{n+1}-|a_n+(a_{n-1}-a_n)x+\dots+ +(a_0-a_1)x^n| \geqslant a_0\rho^{n+1}-\rho^n(a_n+a_{n-1}-a_n+\dots+ +a_0-a_1)=a_0(\rho^{n+1}-\rho^n)>0.$$

Thus, $f(x) \neq 0$ for $|x| > 1$.

748 - 0.6618.

749 2.094551.

750 a) 3.3876, -0.5136, -2.8741;

b) 2.8931;

c) 3.9489, 0.2172, -1.1660;

d) 3.1149, 0.7459, -0.8608.

751 The problem comes down to calculating the root of the equation $x^3 - 3x + 1 = 0$ lying the interval (0, 1).

Answer: $x = 0.347$, to within 0.001.

752 2.4908.

753 a) 1.7320; b) -0.7321; c) 0.6180; d) 0.2679; e) -3.1623; f) 1.2361; g) -2.3028; h) 3.6457; i) 1.6180.

754 a) 1.0953, -0.2624, -1.4773, -2.3556;

b) 0.8270, 0.3383, -1.2090, -2.9563;

c) 1.4689, 0.1168;

d) 8.0060, 1.2855, 0.1960, -1.4875;

e) 1.5357, -0.1537;

f) 3.3322, 1.0947, -0.6002, -1.8268;

g) 0.4910, -1.4910;

h) 2.1462, -0.6821, -1.3178, -4.1463.

CHAPTER VI - SOLUTIONS

SYMMETRIC FUNCTIONS

755 We write out the solution of problem f) in detail:

$$F(x_1, x_2, x_3) = (x_1^2 + x_2^2)(x_1^2 + x_3^2)(x_2^2 + x_3^2).$$

The highest term in the polynomial F is $x_1^4 \cdot x_2^2$.

The homogeneous sixth degree polynomial F can be expressed as the sum of homogeneous sixth degree polynomials made up from

$f_1 = x_1 + x_2 + x_3$

$f_2 = x_1x_2 + x_2x_3 + x_3x_1$

$f_3 = x_1x_2x_3$ in the following combinations:

$f_1f_2f_3$, $f_1^2f_2^2$, $f_1^3f_3$, f_2^3, f_3^2. Thus F can be written:

$F = f_1^2f_2^2 + Af_1^3f_3 + Bf_2^3 + Cf_1f_2f_3 + Df_3^2$, where A, B, C, D are numerical coefficients. To determine the numerical coefficients we give particular values to x_1, x_2, x_3:

x_1	x_2	x_3	f_1	f_2	f_3	F
1	1	0	2	1	0	2
2	—1	—1	0	—3	2	50
1	—2	—2	—3	0	4	200
1	—1	—1	—1	—1	1	8

Thus A, B, C, D must satisfy the system:

$$\begin{aligned} 2 &= 4 + B, \\ 50 &= -27B + 4D, \\ 200 &= -108A + 16D, \\ 8 &= 1 - A - B + C + D, \end{aligned}$$

and so B = -2, D = -1, A = -2, C = 4.

The result is

$$(x_1^2+x_2^2)(x_1^2+x_3^2)(x_2^2+x_3^2)=f_1^2f_2^2-2f_1^3f_3-2f_2^3+4f_1f_2f_3-f_3^2.$$

For the remaining problems we give only the answers:

a) $f_1^3-3f_1f_2$; b) $f_1f_2-3f_3$; c) $f_1^4-4f_1^2f_2+8f_1f_3$;
d) $f_1^3f_2^2-2f_1^4f_3-3f_1f_2^3+6f_1^2f_2f_3+3f_2^2f_3-7f_1f_3^2$;

e) $f_1f_2-f_3$; g) $2f_1^3-9f_1f_2+27f_3$;
h) $f_1^2f_2^2-4f_1^3f_3-4f_2^3+18f_1f_2f_3-27f_3^2$.

756 a) $f_1f_2f_3-f_1^2f_4-f_3^2$; b) $f_1^2f_4+f_3^2-4f_2f_4$;
c) $f_1^3-4f_1f_2+8f_3$.

757 a) $f_1^2-2f_2$; b) $f_1^3-3f_1f_2+3f_3$;
c) $f_1f_3-4f_4$; d) $f_2^2-2f_1f_3+2f_4$;
e) $f_1^2f_2-f_1f_3-2f_2^2+4f_4$; f) $f_1^4-4f_1^2f_2+2f_2^2+4f_1f_3-4f_4$;
g) $f_2f_3-3f_1f_4+5f_5$; h) $f_1^2f_3-2f_2f_3-f_1f_4+5f_5$;
i) $f_1f_2^2-2f_1^2f_3-f_2f_3+5f_1f_4-5f_5$;
j) $f_1^3f_2-3f_1f_2^2-f_1^2f_3+5f_2f_3+f_1f_4-5f_5$;
k) $f_1^5-5f_1^3f_2+5f_1f_2^2+5f_1^2f_3-5f_2f_3-5f_1f_4+5f_5$;
l) $f_2f_4-4f_1f_5+9f_6$; m) $f_3^2-2f_2f_4+2f_1f_5-2f_6$;
n) $f_1^2f_4-2f_2f_4-f_1f_5+6f_6$;
o) $f_1f_2f_3-3f_1^2f_4-3f_3^2+4f_2f_4+7f_1f_5-12f_6$;
p) $f_2^3-3f_1f_2f_3+3f_1^2f_4+3f_3^2-3f_2f_4-3f_1f_5+3f_6$;
q) $f_1^3f_3-3f_1f_2f_3-f_1^2f_4+3f_3^2+2f_2f_4+f_1f_5-6f_6$;
r) $f_1^2f_2^2-2f_1^3f_3-2f_2^3+4f_1f_2f_3+2f_1^2f_4-3f_3^2$
$+2f_2f_4-6f_1f_5+6f_6$;
s) $f_1^4f_2-4f_1^2f_2^2-f_1^3f_3+2f_2^3+7f_1f_2f_3$
$+f_1^2f_4-3f_3^2-6f_2f_4-f_1f_5+6f_6$;
t) $f_1^6-6f_1^4f_2+9f_1^2f_2^2+6f_1^3f_3-2f_2^3-12f_1f_2f_3-6f_1^2f_4$
$+3f_3^2+6f_2f_4+6f_1f_5-6f_6$.

758 a) $nf_1^2-8f_2$;
b) $-f_1^n+4f_1^{n-2}f_2-8f_1^{n-3}f_3+\dots+(-2)^n f_n$.

759 a) $(n-1)f_1^2-2nf_2$; b) $(n-1)f_1^3-3(n-2)f_1f_2+3(n-4)f_3$;
c) $(n-1)f_1^4-4nf_1^2f_2+2(n+6)f_2^2+4(n-3)f_1f_3-4nf_4$;
d) $\frac{3(n-1)(n-2)}{2}f_1^2-(3n-1)(n-2)f_2$.

760 $f_k^2-2f_{k-1}f_{k+1}+2f_{k-2}f_{k+2}-2f_{k-3}f_{k+3}+\dots$

761 $$(n-1)!\sum_{i=1}^n a_i^2f_1^2-2(n-2)!\left[n\sum_{i=1}^n a_i^2-\left(\sum_{i=1}^n a_i\right)^2\right]f_2 = (n-1)!\,S_2s_2+4(n-2)!\,F_2f_2,$$

where $S_2=\sum_{i=1}^n a_i^2$; $s_2=\sum_{i=1}^n x_i^2$; $F_2=\sum_{i<k}a_ia_k$; $f_2=\sum_{i<k}x_ix_k$.

762 a) $\frac{f_1f_2-3f_3}{f_3}$; b) $\frac{2(f_1^2f_2-3f_1f_3-2f_2^2)}{f_1f_2-f_3}$; c) $\frac{f_2^3+f_1^3f_3-6f_1f_2f_3+9f_3^2}{f_3^2}$.

763 a) $\frac{f_2^2-2f_1f_3+2f_4}{f_4}$; b) $\frac{f_1^2f_2^2+f_1^3f_3-6f_1f_2f_3+6f_3^2+2f_1^2f_4}{f_1f_2f_3-f_1^2f_4-f_3^2}$.

764 a) $\frac{f_{n-1}}{f_n}$; b) $\frac{f_{n-1}^2-2f_{n-2}f_n}{f_n^2}$; c) $\frac{f_1f_{n-1}-nf_n}{f_n}$;
d) $\frac{f_1^2f_{n-1}^2-2f_2f_{n-1}^2-2f_1^2f_{n-2}f_n+4f_2f_{n-2}f_n-nf_n^2}{f_n^2}$;
e) $\frac{f_1^2f_{n-1}-2f_2f_{n-1}-f_1f_n}{f_n}$; f) $\frac{f_2f_{n-1}-(n-1)f_1f_n}{f_n}$.

765 -4.

766 -35.

767 16.

768 a) -3; b) $-2p^3-3q^2$; c) $-p^3$ $(x_1^2-x_2x_3=-p)$;
d) q^4; e) $\dfrac{-2p-3q}{1+p-q}$; f) $\dfrac{2p^2-4p-4pq+3q^2+6q}{(1+p-q)^2}$.

769 Let $x_1^2 = x_2^2 + x_3^2$. Then $2x_1^2 = x_1^2 + x_2^2 + x_3^2 = a^2 - 2b$. Thus either $\sqrt{\dfrac{a^2-2b}{2}}$ or $-\sqrt{\dfrac{a^2-2b}{2}}$ is a root of the given equation. Necessary and sufficient conditions that this occur are:

$$a^4(a^2 - 2b) = 2(a^3 - 2ab + 2c)^2.$$

770
$$\begin{aligned} a &= -x_1 - x_2 - x_3, \\ ab - c &= -(x_1+x_2)(x_1+x_3)(x_2+x_3), \\ c &= -x_1x_2x_3. \end{aligned}$$

If all the roots are real and negative then

$$a > 0,\ b > 0,\ c > 0.$$

If one root x_1 is real, and x_2, x_3 are complex conjugates with negative real part, then $x_2 + x_3 < 0$, $x_2x_3 > 0$, $(x_1 + x_2)(x_1 + x_3) > 0$. Thus the conditions $a > 0$, $b > 0$, $c > 0$ again hold.

Now suppose that $a > 0$, $b > 0$, $c > 0$. If x_1 is real and x_2, x_3 are complex conjugates, then $x_2x_3 > 0$, $(x_1 + x_2)(x_1 + x_3) > 0$. Thus, since $c > 0$, $b > 0$, it follows that $x_1 < 0$, $2\,\mathrm{Re}(x_2) = x_2 + x_3 < 0$.

On the other hand if all of x_1, x_2, x_3 are real, then it follows, since $c > 0$, that one of the roots x_1 is negative, and the other two have the same sign. If $x_2 > 0$, $x_3 > 0$, then

$$-x_1 - x_2 > x_3 > 0, \quad -x_1 - x_3 > x_2 > 0.$$

Thus we would have $-(x_1 + x_2)(x_1 + x_3)(x_2 + x_3) < 0$. But this contradicts the hypothesis. Therefore $x_2 < 0$, $x_3 < 0$.

Problem 744 gave another method of solution.

771 $s = \frac{1}{4}\sqrt{a(4ab - a^3 - 8c)}, \quad R = \frac{c}{\sqrt{a(4ab - a^3 - bc)}}.$

772 $a(4ab - a^3 - 8c) = 4c^2.$

773 a) $\frac{25}{27}$; b) $\frac{35}{27}$; c) $-\frac{1679}{625}$.

774 a) $a_1^2a_2^2 - 4a_1^3a_3 - 4a_2^3a_0 + 18a_0a_1a_2a_3 - 27a_0^2a_3^2$;

b) $a_1^3a_3 - a_2^3a_0$; c) $\frac{a_1a_2}{a_0a_3} - 9$; d) $a_1^2a_2^2 - a_1^3a_3 - a_2^3a_0$.

775 It is sufficient to give a proof for the elementary symmetric functions. Let φ_k be an elementary symmetric functions of degree k in $x_2, x_3, \cdots, x_n$; f_k an elementary symmetric function in $x_1, x_2, \cdots, x_n$. Clearly, $\varphi_k = f_k - x_1\varphi_{k-1}$. Thus:

$$\varphi_k = f_k - x_1 f_{k-1} + x_1^2 f_{k-1} - \dots + (-x_1)^{k-1} f_1 + (-1)^k x_1^k,$$
$$(-1)^k \varphi_k = a_k + a_{k-1}x_1 + \dots + a_1 x_1^{k-1} + x_1^k.$$

776
$$x_1+x_2=f_1-x_3;$$
$$(f_1-x_1)(f_1-x_2)(f_1-x_3)=f_1^3-f_1^3+f_1f_2-f_3=f_1f_2-f_3;$$
$$2x_1-x_2-x_3=3x_1-f_1;$$
$$(3x_1-f_1)(3x_2-f_1)(3x_3-f_1)=27f_3-9f_1f_2+2f_1^3;$$
$$x_1^2+x_1x_2+x_2^2=f_1^2-f_2-f_1x_3;$$
$$x_1^2-x_2x_3=f_1x_1-f_2.$$

777 $\sum_{i=1}^{n}\frac{\partial f_k}{\partial x_i}=(n-k)f_{k-1}.$

778 Let $F(x_1, x_2, \ldots, x_n)=\Phi(f_1, f_2, \ldots, f_n)$. Then

$$\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}=n\frac{\partial\Phi}{\partial f_1}+(n-1)f_1\frac{\partial\Phi}{\partial f_2}+\ldots+f_{n-1}\frac{\partial\Phi}{\partial f_n}.$$

779 Let $\varphi(a)=F(x_1+a, x_2+a, \ldots, x_n+a)$. Then

$$\varphi'(a)=\sum_{i=1}^{n}\frac{\partial F(x_1+a, x_2+a, \ldots, x_n+a)}{\partial x_i}.$$

Since $\varphi(a)$ is independent of a, then $\varphi'(a)\equiv 0$, from which $\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}=0$. Conversely, if $\sum_{i=1}^{n}\frac{\partial F(x_1, x_2, \ldots, x_n)}{\partial x_i}\equiv 0$ then

$$\varphi'(a)=\sum_{i=1}^{n}\frac{\partial F(x_1+a, \ldots, x_n+a)}{\partial x_i}=0$$

Therefore $\varphi(a)$ is independent of a; $\varphi(a)=\varphi(0)$, i.e.

$$F(x_1+a, x_2+a, \ldots, x_n+a)=F(x_1, x_2, \ldots, x_n).$$

The result of the preceding problem shows that the relation $\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}=0$ is equivalent to the condition

$$n\frac{\partial\Phi}{\partial f_1}+(n-1)f_1\frac{\partial\Phi}{\partial f_2}+\ldots+f_{n-1}\frac{\partial\Phi}{\partial f_n}=0.$$

780 Let $F(x_1, x_2, \cdots, x_n)$ be a homogeneous symmetric function of degree 2. It must be expressible in terms of the elementary symmetric functions in the form $F = Af_1^2 + Bf_2$. By problem 779, we must have $n2Af_1 + (n-1)Bf_1 = 0$, so that $A = (n-1)\alpha$, $B = -2n\alpha$, and

$$F(x_1, x_2, \ldots, x_n) = \alpha\left[(n-1)f_1^2 - 2nf_2\right] = \alpha\sum_{i<k}(x_i - x_k)^2.$$

781 A symmetric function of the third degree is the following combination of the elementary symmetric functions: $Af_1^3 + Bf_1f_2 + Cf_3$. By problem 779, the relation $3Anf_1^2 + nBf_2 + (n-1)Bf_1^2 + (n-2)Cf_2 = 0$ must hold. Thus

$$F(x_1, x_2, \ldots, x_n) = \alpha\left[(n-1)(n-2)f_1^3 - 3n(n-2)f_1f_2 + 3n^2f_3\right].$$

782 $$(n-2)f_1^2f_2^2 - 2(n-1)f_1^3f_3 - 4(n-2)f_2^3 + (10n-12)f_1f_2f_3 - 4(n-1)f_1^2f_4 - 9nf_3^2 + 8nf_2f_4.$$

783 We may take

$$\varphi_k = f_k\left(x_1 - \frac{f_1}{n},\ x_2 - \frac{f_1}{n},\ \ldots,\ x_n - \frac{f_1}{n}\right).$$

Each of the functions φ_k certainly has the required properties. Moreover if $F(x_1, x_2, \cdots, x_n) = F(x_1 + a, x_2 + a, \cdots, x_n + a)$, and $F(x_1, x_2, \cdots, x_n) = \Phi(f_1, f_2, \ldots, f_n)$, then

$$F(x_1, x_2, \ldots, x_n) = \Phi(0, \varphi_2, \varphi_3, \ldots, \varphi_n).$$

784 a) $-4\varphi_2^3 - 27\varphi_3^2$; b) $18\varphi_2^2$.

785 a) $8\varphi_3$;

b) $-4\varphi_2^3\varphi_3^2+16\varphi_2^4\varphi_4-27\varphi_3^4+144\varphi_2\varphi_3^2\varphi_4-128\varphi_2^2\varphi_4^2+256\varphi_4^3$.

786
$$s_2=f_1^2-2f_2;$$
$$s_3=f_1^3-3f_1f_2+3f_3;$$
$$s_4=f_1^4-4f_1^2f_2+2f_2^2+4f_1f_3-4f_4;$$
$$s_5=f_1^5-5f_1^3f_2+5f_1f_2^2+5f_1^2f_3-5f_2f_3-5f_1f_4+5f_5;$$
$$s_6=f_1^6-6f_1^4f_2+9f_1^2f_2^2+6f_1^3f_3-2f_2^3-12f_1f_2f_3-6f_1^2f_4+3f_3^2+6f_2f_4+6f_1f_5-6f_6.$$

787
$$2f_2=s_1^2-s_2;$$
$$6f_3=s_1^3-3s_1s_2+2s_3;$$
$$24f_4=s_1^4-6s_1^2s_2+8s_1s_3+3s_2^2-6s_4;$$
$$120f_5=s_1^5-10s_1^3s_2+20s_1^2s_3+15s_1s_2^2-20s_2s_3-30s_1s_4+24s_5;$$
$$720f_6=s_1^6-15s_1^4s_2+40s_1^3s_3+45s_1^2s_2^2-120s_1s_2s_3-15s_2^3-90s_1^2s_4+40s_3^2+90s_2s_4+144s_1s_5-120s_6.$$

788 $s_5 = 859$.

789 $s_8 = 13$.

790 $s_{10} = 621$.

791 $s_1 = -1,\ s_2 = s_3 = \cdots = s_n = 0$.

792 Easily established by induction, using the relation
$$as_k+bs_{k-1}+cs_{k-2}=0,$$
where $s_k=x_1^k+x_2^k$.

793 $s_5-s_1^5=5(f_1^2-f_2)(f_3-f_1f_2)$; $s_3-s_1^3=3(f_3-f_1f_2)$.

794 $s_5=-5f_2f_3$; $s_3=3f_3$; $s_2=-2f_2$.

795 $s_7=-7f_2f_5$; $s_2=-2f_2$; $s_5=5f_5$.

796 $x^n-a=0$.

797 $x^n - \frac{a}{1} x^{n-1} + \frac{a^2}{1\cdot 2} x^{n-2} - \dots + (-1)^n \frac{a^n}{n!} = 0.$

798 $x^n + \frac{P_1(\alpha)}{1} x^{n-1} + \frac{P_2(\alpha)}{2!} x^{n-2} + \dots + \frac{P_n(\alpha)}{n!} = 0$, where

$P_1, P_2, \cdots, P_n$ are the Hermite polynomials

$$P_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k e^{-\frac{x^2}{2}}}{dx^k},$$

and α is a root of the Hermite polynomial $P_{n+1}(x)$.

Solution. Suppose the required equation has the form

$$x^n + \alpha x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0.$$

By Newton's formulas,

$$\begin{aligned} a_1 &= \alpha, \\ 2a_2 &= \alpha a_1 - 1, \\ 3a_3 &= \alpha a_2 - a_1, \\ &\cdots\cdots \\ ka_k &= \alpha a_{k-1} - a_{k-2}, \\ &\cdots\cdots \\ na_n &= \alpha a_{n-1} - a_{n-2}, \\ 0 &= \alpha a_n - a_{n-1}, \end{aligned}$$

These relations show that a_k is a polynomial of degree k in α. Set $k!a_k = P_k(\alpha)$. Then, setting $P_0 = 1$, we obtain

$$P_1 = \alpha, \quad P_k - \alpha P_{k-1} + (k-1)P_{k-2} = 0; \quad -\alpha P_n + nP_{n-1} = 0.$$

The first two relations show that P_k is the Hermite polynomial in α (see problem 707). The last relation gives $P_{n+1}(\alpha) = 0$.

799 $\frac{1}{2}\left(s_k^2 - s_{2k}\right)$.

800
$$\sum_{i=1}^{n}(x+x_i)^k = \sum_{m=0}^{k} C_m^k s_{k-m} x^m;$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_j+x_i)^k = \sum_{m=0}^{k} C_m^k s_{k-m} s_m;$$
$$\sum_{i<j}(x_i+x_j)^k = \frac{1}{2}\left(\sum_{m=0}^{k} C_m^k s_{k-m}s_m - 2^k s_k\right).$$

801
$$\sum_{i<j}(x_i-x_j)^{2k} = \frac{1}{2}\sum_{m=0}^{2k} C_m^{2k}(-1)^m s_m s_{2k-m}.$$

802 Add to the first line $-s_1$ times the second plus s_2 times the third, $\cdots$, plus $(-1)^{k-1}s_{k-1}$ times the k-th. By Newton's formula we obtain:

$$\det \begin{bmatrix} f_1 & 1 & 0 \dots & 0 \\ 2f_2 & f_1 & 1 \dots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ (k-1)f_{k-1} & f_{k-2} & \dots f_1 & 1 \\ kf_k & f_{k-1} & \dots & f_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots 0 \\ 0 & f_1 & 1 & \dots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & f_{k-2} & f_{k-3} & \dots 1 \\ (-1)^{k-1}s_k & f_{k-1} & & \dots f_1 \end{bmatrix} = s_k.$$

803 Add to the first column $-f_1$ times the second, f_2 times the third, $\cdots$, $(-1)^{k-1}f_{k-1}$ times the k-th. Use Newton's formula to establish the result.

804 $n!(x^n - f_1x^{n-1} + f_2x^{n-2} + \dots + (-1)^n f_n)$.

805 $\dfrac{\varphi(n)}{\varphi\left(\frac{n}{d}\right)}\mu\left(\frac{n}{d}\right)$, where d = (m,n).

806 By problems 117, 119 it is sufficient to consider the case $n = p_1 p_2 \cdots p_k$ where $p_1, p_2, \cdots, p_k$ are distinct odd prime numbers. In this case $s_1 = s_2 = s_4 = (-1)^k$; $s_3 = 2(-1)^{k-1}$, if $3 | n$; $s_3 = (-1)^k$ if $3 \nmid n$. Calculation with Newton's formula gives:

$$f_2 = \frac{1-(-1)^k}{2} \quad ;$$

$$f_3 = \frac{(-1)^{k-1}-1}{2} \quad , \text{ if } 3 | n;$$

$$f_3 = \frac{(-1)^k - 1}{2} \quad , \text{ if } 3 \nmid n;$$

$$f_4 = \frac{(-1)^{k-1}-1}{2} \quad , \text{ if } 3 | n;$$

$$f_4 = \frac{(-1)^{k-1}+1}{2} \quad , \text{ if } 3 \nmid n.$$

807 $s_1 = s_2 = s_3 = \cdots = s_n = a$. Thus for $k \le n$

$$\begin{aligned} kf_k &= af_{k-1} - af_{k-2} + \dots + (-1)^{k-1} f_1, \\ (k-1) f_{k-1} &= \qquad\qquad af_{k-2} + \dots + (-1)^{k-2} f_1, \end{aligned}$$

from which

$$kf_k = (a-k+1) f_{k-1}, \quad f_k = \frac{a-k+1}{k} f_{k-1}.$$

It is obvious that $f_1 = a$. Therefore

$$f_2 = \frac{a(a-1)}{1 \cdot 2}, \; \dots, \quad f_k = \frac{a(a-1)\dots(a-k+1)}{1 \cdot 2 \dots k}.$$

And thus $x_1, x_2, \cdots, x_n$ are the roots of the equation

$$x^n - \frac{a}{1} x^{n-1} + \frac{a(a-1)}{1 \cdot 2} x^{n-2} - \dots + (-1)^n \frac{a(a-1)\dots(a-n+1)}{n!} = 0;$$

$$s_{n+1} = a - \frac{a(1-a)(2-a)\dots(n-a)}{n!}.$$

808 $$(x-a)(x-b)[x^n+(a+b)x^{n-1}+\ldots+(a^n+a^{n-1}b+\ldots$$
$$\ldots+b^n)]=(x-a)[x^{n+1}+ax^n+a^2x^{n-1}+\ldots+a^nx$$
$$-b(a^n+a^{n-1}b+\ldots+b^n)]=x^{n+2}-(a^{n+1}+a^nb+\ldots+b^{n+1})x$$
$$+ab(a^n+a^{n+1}+\ldots+b^n).$$

In the new polynomials, the degrees of the sums $\sigma_1, \sigma_2, \cdots, \sigma_n$ are obviously zero. But $\sigma_k = s_k + a^k + b^k$. Therefore $s_k = -(a^k + b^k)$ for $1 \leq k \leq n$.

809 $s_k = -a^k - b^k$, if k is odd;

$s_k=-\left(a^{\frac{k}{2}}-b^{\frac{k}{2}}\right)^2$, if k is even.

810 a) $(x+a)(x^2+ax+b)-c=0;$
b) $x(x-a^2+3b)^2-(a^2b^2-4a^3c-4b^3+18abc-27c^2)=0;$
c) $x^3+(3b-a^2)x^2+b(3b-a^2)x+b^3-a^3c=0;$
d) $x^2(x-a^2+3b)+(a^2b^2-4a^3c-4b^3+18abc-27c^2)=0;$
e) $x^3-(a^2-2b)x^2+(b^2-2ac)x-c^2=0;$
f) $x^3+(a^3-3ab+3c)x^2+(b^3-3abc+3c^2)x+c^3=0.$

811 $y^2+(2a^3-9ab+27c)y+(a^2-3b)^3=0.$

812 $y^2-\frac{ab-3c}{c}y+\frac{b^3+a^3c-6abc+9c^2}{c^2}=0.$

813 $$y^6-\frac{ab-3c}{c}y^5+\frac{b^3-5abc+6c^2}{c^2}y^4-$$
$$-\frac{a^2b^2-2b^3-2a^3b+6abc-7c^2}{c^2}y^3+\frac{b^3-5abc+6c^2}{c^2}y^2-\frac{ab-3c}{c}y+1=0.$$

814 a) $y^3 - by^2 + (ac - 4d)y - (a^2d + c^2 - 4bd) = 0$ (Ferrari resolvent);

b) $y^3 - (3a^2 - 8b)y^2 + (3a^4 - 16a^2b + 16b^2$
$+ 16ac - 64d)y - (a^3 - 4ab + 8c)^2 = 0$
(Ferrari resolvent);

c) $y^6 - by^5 + (ac - d)y^4 - (a^2d + c^2 - 2bd)y^3$
$+ d(ac - d)y^2 - bd^2y + d^3 = 0;$

d) $y^6 + 3ay^5 + (3a^2 + 2b)y^4 + (a^3 + 4ab)y^3$
$+ (2a^2b + b^2 + ac - 4d)y^2 + (ab^2 + a^2c - 4ad)y$
$+ (abc - a^2d - c^2) = 0.$

815 $$x = \frac{-a \pm \sqrt{a^2 - 4b + 4y_1} \pm \sqrt{a^2 - 4b + 4y_2} \pm \sqrt{a^2 - 4b + 4y_3}}{4}.$$

The ambiguous signs must be chosen so that the product of the square roots is $-a^3 + 4ab - 8c$.

816 $$x = \frac{\pm\sqrt{4a + \sqrt[3]{b^2 - 64a^3}} \pm \sqrt{4a + \varepsilon\sqrt[3]{b^2 - 64a^3}} \pm \sqrt{4a + \varepsilon^2\sqrt[3]{b^2 - 64a^3}}}{2},$$

$\varepsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$. Choose the ambiguous signs so that the product of the roots is $-b$.

817 $(y + a)^4(y^2 + 6ay + 25a^2) + 3125b^4y = 0.$

Method. The roots of the equation we are seeking must be:

$$y_1 = (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) \times (x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1);$$
$$y_2 = (x_1x_3 + x_3x_2 + x_2x_5 + x_5x_4 + x_4x_1) \times (x_1x_2 + x_2x_4 + x_4x_3 + x_3x_5 + x_5x_1);$$
$$y_3 = (x_5x_2 + x_2x_4 + x_4x_3 + x_3x_1 + x_1x_5) \times (x_5x_4 + x_4x_1 + x_1x_2 + x_2x_3 + x_3x_5);$$
$$y_4 = (x_2x_1 + x_1x_3 + x_3x_5 + x_5x_4 + x_4x_2) \times (x_2x_3 + x_3x_4 + x_4x_1 + x_1x_5 + x_5x_2);$$
$$y_5 = (x_5x_3 + x_3x_2 + x_2x_4 + x_4x_1 + x_1x_5) \times (x_5x_2 + x_2x_1 + x_1x_3 + x_3x_4 + x_4x_5);$$
$$y_6 = (x_2x_1 + x_1x_4 + x_4x_3 + x_3x_5 + x_5x_2) \times (x_2x_4 + x_4x_5 + x_5x_1 + x_1x_3 + x_3x_2).$$

The the equation we are looking for must have the form

$$y^6+c_1ay^5+c_2a^2y^4+c_3a^3y^3+c_4a^4y^2+(c_5a^5+c_6b^4)y + (c_7a^6+c_8ab^4)=0,$$

where $c_1, c_2, \cdots, c_8$ are absolute constants. To determine them we set a = -1, b = 0; a = 0, b = -1. We obtain the following table.

a	b	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	y_6
-1	0	1	i	-1	$-i$	0	1	$3-4i$	1	1	$3+4i$	1
0	-1	1	ε	ε^2	ε^3	ε^4	0	-5	$-5\varepsilon^4$	$-5\varepsilon^3$	$-5\varepsilon^2$	-5ε

In the first case the equation we seek must have the form:

$$(y-1)^4(y^2-6y+25)=0 \ ;$$

in the second case $y^6+3125y=0$. In this way we have determined all the coefficients except c_8. To show that $c_8 = 0$, we can take for example a = -5, b = 4. In this case $x_1 = x_2 = 1$, and the remaining roots satisfy the equation $x^3 + 2x^2 + 3x + 4 = 0$, and all the required calculations can be completed without difficulty.

818 Set $f(x) = (x - x_1)(x - x_2)\cdots(x - x_n)$, where $x_1, x_2, \cdots, x_n$ are independent variables. Now set

$$x^{k-1}\varphi(x)=f(x)q_k(x)+r_k(x) \ ,$$

$$r_k(x) = c_{k1} + c_{k2}x + \dots + c_{kn}x^{n-1}.$$

It is obvious that the coefficients $c_{k,s}$ are certain polynomials in $x_1, x_2, \cdots, x_n$. Therefore,

$$\det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} =$$

$$= \det \begin{bmatrix} r_1(x_1) & r_1(x_2) & \dots & r_1(x_n) \\ r_2(x_1) & r_2(x_2) & \dots & r_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ r_n(x_1) & r_n(x_2) & \dots & r_n(x_n) \end{bmatrix} =$$

$$= \det \begin{bmatrix} \varphi(x_1) & \varphi(x_2) & \dots & \varphi(x_n) \\ x_1\varphi(x_1) & x_2\varphi(x_2) & \dots & x_n\varphi(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1}\varphi(x_1) & x_2^{n-1}\varphi(x_2) & \dots & x_1^{n-1}\varphi(x_n) \end{bmatrix} =$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n) \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$

so that

$$\det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \varphi(x_1)\varphi(x_2)\dots\varphi(x_n) = R(f, \varphi).$$

This last equation is an identity among polynomials in the independent variables $x_1, x_2, \cdots, x_n$ and therefore is valid for every numerical value of these variables.

819 First note that every polynomial $\psi_k(x)$ has degree n - 1. Use the following abbreviation:

$$\begin{aligned}
f_k(x) &= a_0x^{k-1} + \dots + a_{k-1};\\
\bar{f}_k(x) &= a_kx^{n-k} + \dots + a_n;\\
\varphi_k(x) &= b_0x^{k-1} + \dots + b_{k-1};\\
\bar{\varphi}_k(x) &= b_kx^{n-k} + \dots + b_n.
\end{aligned}$$

Then

$$f(x) = x^{n-k+1}f_k(x) + \bar{f}_k(x);$$
$$\varphi(x) = x^{n-k+1}\varphi_k(x) + \bar{\varphi}_k(x);$$
$$\psi_k(x) = f_k(x)[x^{n-k+1}\varphi_k(x) + \bar{\varphi}_k(x)] -$$
$$- \varphi_k(x)[x^{n-k+1}f_k(x) + \bar{f}_k(x)] = f_k(x)\bar{\varphi}_k - \varphi_k\bar{f}_k(x) =$$
$$= (a_0b_k - b_0a_k)x^{n-1} + \dots$$

Set $\psi_k(x) = c_{k1} + c_{k2}x + \dots + c_{kn}x^{n-1}$ and let $x_1, x_2, \cdots, x_n$ be the roots of f(x). Then

$$\det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} =$$

$$= \det \begin{bmatrix} \psi_1(x_1) & \psi_1(x_2) & \dots & \psi_1(x_n) \\ \psi_2(x_1) & \psi_2(x_2) & \dots & \psi_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \psi_n(x_1) & \psi_n(x_2) & \dots & \psi_n(x_n) \end{bmatrix} =$$

$$= \det \begin{bmatrix} f_1(x_1)\varphi(x_1) & f_1(x_2)\varphi(x_2) & \dots & f_1(x_n)\varphi(x_n) \\ f_2(x_1)\varphi(x_1) & f_2(x_2)\varphi(x_2) & \dots & f_2(x_n)\varphi(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ f_n(x_1)\varphi(x_1) & f_n(x_2)\varphi(x_2) & \dots & f_n(x_n)\varphi(x_n) \end{bmatrix} =$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n) \cdot \begin{vmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_n) \end{vmatrix} =$$

$$= \varphi(x_1)\varphi(x_2)\ \dots\ \varphi(x_n)$$

$$\times\ \det \begin{bmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} =$$

$$= \varphi(x_1)\varphi(x_2)\ \dots\ \varphi(x_n)\, a_0^n \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$

Thus we obtain

$$\det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = a_0^n \varphi(x_1)\varphi(x_2)\ \dots\ \varphi(x_n) = R(f,\ \varphi).$$

820 The polynomials χ_k have degree n - 1 or less. This is clear for $1 \le k \le n - m$, and for $k > n - m$, it follows from the fact that χ_k is the Bezout polynomial ψ_{k-n+m} of $f(x)$, $x^{n-m}\varphi(x)$. Set $\chi_k(x) = c_{k1} + c_{k2}x + \dots + c_{kn}x^{n-1}$;

$$\Delta = \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$

Then

$$\Delta = \det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \det \begin{bmatrix} \chi_1(x_1) & \chi_1(x_2) & \dots & \chi_1(x_n) \\ \chi_2(x_1) & \chi_2(x_2) & \dots & \chi_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \chi_n(x_1) & \chi_n(x_2) & \dots & \chi_n(x_n) \end{bmatrix} =$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)$$

$$\det \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-m-1} & x_2^{n-m-1} & \dots & x_n^{n-m-1} \\ x_1^{n-m} f_0(x_1) & x_2^{n-m} f_0(x_2) & \dots & x_n^{n-m} f_0(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-m} f_{m-1}(x_1) & x_2^{n-m} f_{m-1}(x_2) & \dots & x_n^{n-m} f_{m-1}(x_n) \end{bmatrix} =$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\Delta \cdot \begin{vmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & a_0 & & & \\ & & & & a_1 & a_0 & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & a_{m-1} & a_{m-2} & \dots & a_0 \end{vmatrix} =$$

$$= a_0^m \varphi(x_1)\dots\varphi(x_n)\Delta,$$

from which the required result follows.

821 a) -7; b) 243; c) 0; d) -59; e) 4854; f) $(b_0a_2-b_2a_0)^2-$
$-(b_0a_1-b_1a_0)(b_1a_2-b_2a_1)$.

822 a) For $\lambda = 3$ and $\lambda = -1$;

b) $\lambda=1,\quad \lambda=\dfrac{-2+\sqrt{2}\pm\sqrt{4\sqrt{2}-2}}{2}$,

$\lambda=\dfrac{-2-\sqrt{2}\pm i\sqrt{4\sqrt{2}+2}}{2}$;

c) $\lambda=\pm\sqrt{-2},\ \lambda=\pm\sqrt{-12}$.

823 a) $y^6-4y^4+3y^2-12y+12=0$;

b) $5y^5-7y^4+6y^3-2y^2-y-1=0$;

c) $y^3+4y^2-y-4=0$.

824 a) $x_1=1,\quad x_2=2,\quad x_3=0,\quad x_4=-2,$
$y_1=2;\quad y_2=3;\quad y_3=-1;\quad y_4=1.$

b) $x_1=0,\quad x_2=3,\quad x_3=2,\quad x_4=2,$
$y_1=1;\quad y_2=0;\quad y_3=2;\quad y_4=-1.$

c) $x_1=1,\quad x_2=1,\quad x_3=-1,\quad x_4=2,$
$y_1=-1;\quad y_2=-1;\quad y_3=1;\quad y_4=2.$

d) $x_1=0,\quad x_2=0,\quad x_3=1,\quad x_4=1,\quad x_{5,6}=2,$
$y_1=1;\quad y_2=3;\quad y_3=2;\quad y_4=3;\quad y_{5,6}=1\pm i\sqrt{2}.$

e) $x_1=0,\quad x_2=0,\quad x_3=2,\quad x_4=x_5=2,\quad x_6=-4,$
$y_1=2;\quad y_2=-2;\quad y_3=0;\quad y_4=y_5=2;\quad y_6=2;$
$x_7=4,\quad x_8=-6,\quad x_9=-2/3,$
$y_7=6;\quad y_8=4;\quad y_9=4/3.$

825 $a_0^na_n^{n-1}$.

826 Set $f(x)=a_0(x-x_1)(x-x_2)\dots(x-x_n)$;
$\varphi_1(x)=b_0x^k+\dots+b_k;\quad \varphi_2(x)=c_0x^m+\dots+c_m.$

Then

$$R(f,\ \varphi_1\cdot\varphi_2)=a_0^{m+k}\prod_{i=1}^{n}\varphi_1(x_i)\varphi_2(x_i)=$$
$$=\left[a_0^m\prod_{i=1}^{n}\varphi_1(x_i)\right]\left[a_0^k\prod_{i=1}^{n}\varphi_2(x_i)\right]=R(f,\ \varphi_1)\cdot R(f,\ \varphi_2).$$

827 Only the case $n > 2$ is of interest. Set $d = (m,n)$; let $\xi_1, \xi_2, \cdots$ be the primitive n-th roots of unity; $\eta_1, \eta_2, \cdots$ be the primitive n/d-th roots of unity; $n_1 = n/d$. Then

$$R(X_n,\ x^m-1)=\prod(\xi_i^m-1)=\prod(1-\xi_i^m)=$$
$$=\left[\prod(1-\eta_i)\right]^{\frac{\varphi(n)}{\varphi(n_1)}}[X_{n_1}(1)]^{\frac{\varphi(n)}{\varphi(n_1)}}.$$

If m is divisible by n, then $R(X_n, x^m - 1) = 0$. If m is not divisible by n, then $n_1 \neq 1$, and by problem 123, $X_{n_1}(1) = 1$ for $X_{n_1}(1)=1$; $n_1\neq p^\lambda$, $X_{n_1}(1) = p$ for $n_1=p^\lambda$, that is for n_1 a prime power. Thus

$$R(X_n,\ x^m-1)=0 \quad \text{for } n_1 = n/d = 1;$$

$$R(X_n,\ x^m-1)=p^{\frac{\varphi(n)}{\varphi(n_1)}} \quad \text{for } n_1 = n/d = p^\lambda;$$

$$R(X_n,\ x^m-1)=1 \quad \text{otherwise}.$$

828 It is clear that $R(X_n, X_m)$ is a positive integer dividing $R(X_n, x^m - 1)$ and $R(X_m, x^n - 1)$. Set $d = (m,n)$. First suppose $m\nmid n$, $n\nmid m$. Then m/d, n/d are different from 1 and are relatively

prime. By the preceding problem $R(X_n, x^m - 1)$, $R(X_m, x^n - 1)$ are relatively prime and therefore $R(X_n, X_m) = 1$.

Now suppose that $m|n$. If $m = n$, then $R(X_m, X_n) = 0$. If m/n is not a prime power, then $R(X_m, x^n - 1) = 1$, and therefore $R(X_m, X_n) = 1$. Finally suppose that $m = np^\lambda$. Then

$$R(X_m, X_n) = \prod_{\delta/n} R(X_m, x^\delta - 1)^{\mu\left(\frac{n}{\delta}\right)}.$$

All the factors in this formula are 1 except for those for which m/δ is a power of the prime p.

If n is not divisible by p, then a single factor (the one for $\delta = n$) is different from 1, and

$$R(X_m, X_n) = R(X_m, x^n - 1) = p^{\frac{\varphi(m)}{\varphi(m/n)}} = p^{\varphi(n)}.$$

If n is divisible by p, then there are two factors different from 1: the factor for $\delta = n$, and the factor for $\delta = n/p$. Then

$$R(X_m, X_n) = \frac{R(X_m, x^n - 1)}{R(X_m, x^{n/p} - 1)} = p^{\frac{\varphi(m)}{\varphi(m/n)} - \frac{\varphi(m)}{\varphi(mp/n)}} =$$

$$= p^{\varphi(m)\left[\frac{1}{p^{\lambda-1}(p-1)} - \frac{1}{p^\lambda(p-1)}\right]} = p^{\frac{\varphi(m)}{p^\lambda}} = p^{\varphi(n)}.$$

Thus

$R(X_m, X_n) = 0$ for $m = n$;

$R(X_m, X_n) = p^{\varphi(n)}$ for $m = np^\lambda$;

$R(X_m, X_n) = 1$ otherwise.

829 a) 49; b) -107; c) -843; d) 725; e) 2777.

830 a) $3125(b^2-4a^5)^2$; b) $\lambda^4(4\lambda-27)^3$;
c) $(b^2-3ab+9a^2)^2$; d) $4(\lambda^2-8\lambda+32)^3$.

831 a) $\lambda=\pm 2$; b) $\lambda_1=3$, $\lambda_{2,3}=3\left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)$;
c) $\lambda_1=0$, $\lambda_2=-3$, $\lambda_3=125$;
d) $\lambda_1=-1$, $\lambda_2=-\frac{3}{2}$, $\lambda_{3,4}=\frac{7}{2}\pm\frac{9}{2}i\sqrt{3}$.

832 In general if the discriminant is positive, the number of pairs of conjugate complex roots is even; if the discriminant is negative, the number of such pairs is odd.

In particular for a third degree polynomial all roots are real if $D > 0$; there are two complex conjugate roots if $D < 0$.

If $D > 0$ for a fourth degree polynomial, then either all roots are real or all roots are complex. For $D < 0$ in this case, there are two real roots and a single pair of conjugate complex roots.

833 $f=x^n+a$; $f'=nx^{n-1}$;
$$R(f',f)=n^n a^{n-1};\quad D(f)=(-1)^{\frac{n(n-1)}{2}}n^n a^{n-1}.$$

834 $f=x^n+px+q$; $f'=nx^{n-1}+p$;
$$R(f',f)=n^n\prod_{k=0}^{n-2}\left(q+\frac{n-1}{n}p\sqrt[n-1]{-\frac{p}{n}}\varepsilon^k\right),$$
where $\varepsilon=\cos\frac{2\pi}{n-1}+i\sin\frac{2\pi}{n-1}$.

$$R(f',\ f)=n^n\left[q^{n-1}+\frac{(n-1)^{n-1}p^{n-1}}{n^{n-1}}\cdot\left(-\frac{p}{n}\right)(-1)^{n-2}\right]=$$
$$=n^nq^{n-1}+(-1)^{n-1}(n-1)^{n-1}p^n;$$
$$D(f)=(-1)^{\frac{n(n-1)}{2}}n^nq^{n-1}+(-1)^{\frac{(n-1)(n-2)}{2}}(n-1)^{n-1}p^n.$$

835 Set d = (m,n). Set $m_1 = m/d$, $n_1 = n/d$, and let ε be a primitive n-th root of 1, η a primitive n_1-th root of 1, $f(x) = a_0x^{m+n}+a_1x^m+a_2$.
Then $f'(x)=(m+n)a_0x^{m+n-1}+ma_1x^{m-1}$.
The roots of this derivative are: $\xi_1=\xi_2=\ldots=\xi_{m-1}=0$;

$$\xi_{m+k}=\sqrt{-\frac{ma_1}{(m+n)a_0}}\,\varepsilon^k=\xi_m\varepsilon^k,\quad k=0,\ 1,\ \ldots,\ n-1.$$

Thus

$$R(f',\ f)=(m+n)^{m+n}a_0^{m+n}a_2^{m-1}\prod_{k=0}^{n-1}\left[a_2+\frac{na_1}{m+n}\xi_m^m\varepsilon^{km}\right]=$$
$$=(m+n)^{m+n}a_0^{m+n}a_2^{m-1}\left[\prod_{k=0}^{n_1-1}\left(a_2+\frac{na_1}{m+n}\right)\xi_m\eta^k\right]^d=$$
$$=(m+n)^{m+n}a_0^{m+n}a_2^{m-1}\left[a_2^{n_1}+(-1)^{m_1+n_1-1}\frac{n^{n_1}m^{m_1}a_1^{m_1+n_1}}{(m+n)^{m_1+n_1}a_0^{m_1}}\right]^d=$$
$$=a_0^na_2^{m-1}\left[(m+n)^{m_1+n_1}a_0^{m_1}a_2^{n_1}+(-1)^{m_1+n_1-1}n^{n_1}m^{m_1}a_1^{m_1+n_1}\right]^d$$

and therefore

$$D(f)=(-1)^{\frac{(m+n)(m+n-1)}{2}}a_0^{n-1}a_2^{m-1}\left[(m+n)^{m_1+n_1}a_0^{m_1}a_2^{n_1}+\right.$$
$$\left.+(-1)^{m_1+n_1-1}n^{n_1}m^{m_1}a_1^{m_1+n_1}\right]^d.$$

836 The discriminants are equal.

837 $x_1x_2+x_3x_4-x_1x_3-x_2x_4=(x_1-x_4)(x_2-x_3);$
$x_1x_3+x_2x_4-x_1x_4-x_2x_3=(x_1-x_2)(x_3-x_4);$
$x_1x_4+x_2x_3-x_1x_2-x_3x_4=(x_1-x_3)(x_4-x_2).$

To obtain the desired conclusion, we square each of these formulas and multiply the results.

838 Set $f(x)=a_0(x-x_1)(x-x_2)\dots(x-x_n)$. Then

$$D(f(x)(x-a))=a_0^{2n}(a-x_1)^2(a-x_2)^2\dots(a-x_n)^2\prod_{i<k}(x_i-x_k)^2 = D(f(x))[f(a)]^2.$$

839 Set $\varphi(x)=x^{n-1}+x^{n-2}+\dots+1$. Then $(x-1)\varphi(x)=x^n-1$, so that

$$D(\varphi)[\varphi(1)]^2=D(x^n-1)=(-1)^{\frac{(n-1)(n-2)}{2}}n^n.$$

Therefore

$$D(\varphi)=(-1)^{\frac{(n-1)(n-2)}{2}}n^{n-2}.$$

840 Set $\varphi(x)=x^n+ax^{n-1}+ax^{n-2}+\dots+a$. Then $\varphi(x)(x-1)=x^{n+1}+(a-1)x^n-a$. Thus

$$(na+1)^2D(\varphi)=(-1)^{\frac{n(n-1)}{2}}a^{n-1}[(n+1)^{n+1}a+n^n(1-a)^{n+1}].$$

Finally

$$D(\varphi)=(-1)^{\frac{n(n-1)}{2}}a^{n-1}\frac{(n+1)^{n+1}a+n^n(1-a)^{n+1}}{(1+na)^2}.$$

841 Set $f(x)=a_0(x-x_1)(x-x_2)\dots(x-x_n)$,
$\varphi(x)=b_0(x-y_1)(x-y_2)\dots(x-y_m)$.

Then

$$D(f\varphi)=(a_0b_0)^{2m+2n-2}\prod_{i<k}(x_i-x_k)^2\prod_{i<k}(y_i-y_k)^2\prod_{i=1}^{n}\prod_{k=1}^{m}(x_i-y_k)^2=$$
$$=a_0^{2n-2}\prod_{i<k}(x_i-x_k)^2b_0^{2m-2}\times$$
$$\times\prod_{i<k}(y_i-y_k)^2\left[a_0^mb_0^n\prod_{i=1}^{n}\prod_{k=1}^{m}(x_i-y_k)\right]^2=D(f)D(\varphi)[R(f,\ \varphi)]^2.$$

842 $X_{p^m}(x^{p^{m-1}}-1)=x^{p^m}-1$. Thus

$$D(X_{p^m})D(x^{p^{m-1}}-1)[R(x^{p^{m-1}}-1,\ X_{p^m})]^2=D(x^{p^m}-1).$$

Using known values in this relationship, we obtain

$$D\left(X_{p^m}\right) = p^{mp^m-(m+1)p^{m-1}}(-1)^{\frac{1}{2}p^{m-1}(p-1)}.$$

843 $$X_n \prod_{\delta/n} (x^\delta - 1)^{\mu\left(\frac{n}{\delta}\right)} = (x^n - 1) \prod_{\substack{\delta/n \\ \delta \neq n}} (x^\delta - 1)^{\mu\left(\frac{n}{\delta}\right)}.$$

Let ε be a root of X_n. Then

$$X'_n(\varepsilon) = n\varepsilon^{n-1} \prod_{\substack{\delta/n \\ \delta \neq n}} (\varepsilon^\delta - 1)^{\mu\left(\frac{n}{\delta}\right)}.$$

The calculation will be simplified if we take only the absolute value of the discriminant of X_n:

$$|D(X_n)| = \prod_{\varepsilon} |X'_n(\varepsilon)| = n^{\varphi(n)} \prod_{\substack{\delta/n \\ \delta \neq n}} \prod_{\varepsilon} (1 - \varepsilon^\delta)^{\mu\left(\frac{n}{\delta}\right)} =$$

$$= n^{\varphi(n)} \prod_{\substack{\delta/n \\ \delta \neq n}} \left[X_{\frac{n}{\delta}}(1)\right]^{\frac{\varphi(n)}{\varphi\left(\frac{n}{\delta}\right)}\mu\left(\frac{n}{\delta}\right)}.$$

Now, $X_{\frac{n}{\delta}}(1) \neq 1$ only if n/δ is a power of some prime. On the other hand $\mu(n/\delta) \neq 0$ only if n/δ is not the square of a prime number. Therefore the only factors to be retained are those corresponding to $n/\delta = p_1, p_2, \cdots, p_k$, where $p_1, p_2, \cdots, p_k$ are the distinct prime divisors of n.

Thus

$$|D(X_n)| = \frac{n^{\varphi(n)}}{\prod_{p/n} p^{\varphi(n)/p-1}}.$$

Since all the roots of X_n are complex, the sign of its discriminant will be $(-1)^{\frac{\varphi(n)}{2}}$.

Thus

$$D(X_n) = (-1)^{\frac{\varphi(n)}{2}} \frac{n^{\varphi(n)}}{\prod_{p/n} p^{\varphi(n)/p-1}}.$$

844 $$E_n=n!\left(1+\frac{x}{1}+\dots+\frac{x^n}{n!}\right),$$
$$E_n'=n!\left(1+\frac{x}{1}+\dots+\frac{x^{n-1}}{(n-1)!}\right).$$

Thus

$$E_n'=E_n-x^n;$$
$$R(E_n,\ E_n')=\prod_{i=1}^{n}(-x_i)^n=(-1)[(-1)^n n!]^n=(n!)^n;$$
$$D(E_n)=(-1)^{\frac{n(n-1)}{2}}(n!)^n.$$

845 It is easy to see that

$$(nx+n-a)F_n-x(x+1)F_n'+\frac{a(a-1)\dots(a-n)}{n!}=0.$$

Let $x_1,\ x_2,\cdots,x_n$ be the roots of F_n. Then

$$F_n'(x_i)=\frac{c}{x_i(x_i+1)}\quad,\text{ where }\quad c=\frac{a(a-1)\dots(a-n)}{n!}.$$

Thus

$$R(F_n,\ F_n')=\frac{c^n}{\Pi x_i\Pi(x_i+1)}=\frac{c^n}{\frac{a(a-1)\dots(a-n+1)}{n!}\cdot\frac{(a-1)\dots(a-n)}{n!}}=$$
$$=\frac{a^{n-1}(a-1)^{n-2}(a-2)^{n-2}\dots(a-n+1)^{n-2}(a-n)^{n-1}}{(n!)^{n-2}};$$
$$D(F_n)=(-1)^{\frac{n(n-1)}{2}}\frac{a^{n-1}(a-1)^{n-2}(a-2)^{n-2}\dots(a-n+1)^{n-2}(a-n)^{n-1}}{(n!)^{n-2}}.$$

846 $P_n'=nP_{n-1}$. Thus,

$$R(P_n,\ P_n')=n^nR(P_n,\ P_{n-1}).$$

and

$$P_n-xP_{n-1}+(n-1)P_{n-2}=0.$$

Therefore $P_n(\xi) = -(n-1)P_{n-2}(\xi)$, if ξ is a root of P_{n-1}, and therefore

$$R(P_n, P_{n-1}) = (-1)^{n-1}(n-1)^{n-1}R(P_{n-2}, P_{n-1}) = \\ = (-1)^{n-1}(n-1)^{n-1}R(P_{n-1}, P_{n-2}).$$

Now it is easy to see that

$$R(P_n, P_{n-1}) = (-1)^{\frac{n(n-1)}{2}}(n-1)^{n-1}(n-2)^{n-2}\dots 2^2\cdot 1.$$

Finally

$$D(P_n) = 1\cdot 2^2\cdot 3^3\dots (n-1)^{n-1}n^n.$$

847. $D(P_n) = 1\cdot 2^3\cdot 3^5\dots n^{2n-1}$.
848. $D(P_n) = 2^{n-1}n^n$.
849. $D(P_n) = (n+1)^{n-1}\cdot 2^{n(n-1)}$.
850. $D(P_n) = 1\cdot 2^3\cdot 3^5\dots n^{2n-1}\cdot 1^{2(n-1)}\cdot 3^{2(n-2)}\dots (2n-3)^2$.
851. $D(P_n) = 2^2\cdot 3^4\dots n^{2n-2}\cdot (n+1)^{n-1}$.

852 Set $f(x) = x^n + a_1x^{n-1} + \dots + a_n =$
$$= (x-x_1)(x-x_2)\dots(x-x_n),$$

$D(f) = \Pi(x_i - x_k)^2$. We obtain the maximum of D(f) by solving the following system of equations

$$x_1^2 + x_2^2 + \dots + x_n^2 = n(n-1)R^2,$$
$$\frac{\partial}{\partial x_i}\left(D - \lambda(x_1^2 + x_2^2 + \dots + x_n^2)\right) = 0.$$

It is easy to see that

$$\frac{\partial D}{\partial x_i} = \frac{Df''(x_i)}{f'(x_i)},$$

Thus we have

$$f''(x)D - 2\lambda x_i f'(x_i) = 0 \quad \text{for } i = 1, 2, \dots, n.$$

Therefore the polynomial f(x) that has maximum discriminant must satisfy the differential

equation

$$cf(x) - 2\lambda x f'(x) + Df''(x) = 0,$$

where c is some constant. Dividing this by c/n and comparing coefficients of x^n, we see that the differential equation must have the form

$$nf(x) - xf'(x) + c'f''(x) = 0,$$

where c' is some new constant.

Using the method of undetermined coefficients for the next highest and next next highest powers of x, we find $a_1 = 0$, $a_2 = -n(n-1)c'/2$. Now we can determine c'. Indeed

$$n(n-1)R^2 = x_1^2 + x_2^2 + \ldots + x_n^2 = a_1^2 - 2a_2 = n(n-1)c'.$$

Thus $c' = R^2$.

Computing the remaining coefficients we find that f(x) has the form

$$f(x) = x^n - \frac{n(n-1)}{2}R^2x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4}R^4x^{n-4} - \ldots$$

It is easy to see that

$$f(x) = R^nP_n\left(\frac{x}{R}\right),$$

where P_n is the Hermite polynomial. The discriminant is given by

$$D(f) = R^{n(n-1)}\cdot 1\cdot 2^2\cdot 3^3\ldots n^n.$$

This is the required maximum discriminant.

853 $2^{2n}(-1)^n a_0a_n[D(f)]^2$.

854 $m^{mn}(-1)^{\frac{m(m-1)n}{2}}a_0^{m-1}a_n^{m-1}[D(f)]^m$.

855 $$F(x)=\prod_{i=1}^{n}(\varphi(x)-x_i).$$

Thus

$$D(F)=\prod_{i=1}^{n}D(\varphi(x)-x_i)\left[\prod_{i<k}R(\varphi(x)-x_i,\ \varphi(x)-x_k)\right]^2.$$

It is clear therefore, that

$$R(\varphi(x)-x_i,\ \varphi(x)-x_k)=(x_i-x_k)^m.$$

Therefore

$$D(F)=\prod_{i=1}^{n}D(\varphi(x)-x_i)\prod_{i<k}(x_i-x_k)^{2m}=[D(f)]^m\prod_{i=1}^{n}D(\varphi(x)-x_i),$$

which established the assertion.

856 (y + 1)(y = 5)(y - 19) = 0.

857 a) Solution. $x^3 = 3x + 4$. Set $y = 1 + x + x^2$, where x is a root of the given equation. Then

$$yx=x+x^2+x^3=x+x^2+3x+4=4+4x+x^2;$$
$$yx^2=4x+4x^2+x^3=4x+4x^2+3x+4=4+7x+4x^2.$$

Eliminating x, we obtain

$$\det\begin{bmatrix}1-y & 1 & 1\\ 4 & 4-y & 1\\ 4 & 7 & 4-y\end{bmatrix}=0,$$

$$y^3-9y^2+9y-9=0;$$

b) $y^3-7y^2+3y-1=0;$

c) $y^4+5y^3+9y^2+7y-6=0;$

d) $y^4-12y^3+43y^2-49y+20=0.$

858 a) $y^3-2y^2+6y-4=0,\ x=-\dfrac{y^2-2y+4}{2};$

b) $y^4-9y^3+31y^2-45y+13=0,\ x=\dfrac{y^2-3y+2}{3};$

c) $y^4+2y^3-y^2-2y+1=0$; no inverse transformation.

859 $y^3 - y^2 - 2y + 1 = 0.$

The transformed equation is the same as the original. This means that the original equation happens to have roots x_1, x_2, connected by the relation $x_2 = 2 - x_1^2$.

860 x_2 is expressible rationally in terms of x_1. Since x_1^3 and higher powers of x_1 can be expressed rationally in terms of lower powers, we may write $x_2 = ax_1^2 + bx_1 + c$. The numbers $ax_1^2 + bx_1 + c$, $ax_2^2 + bx_2 + c$, $ax_3^2 + bx_3 + c$ are roots of some cubic equation with rational coefficients. One of these is a root of the given equation. All are rational, and the given equation is irreducible. Thus all three are roots of the given equation. In other words either $ax_2^2 + bx_2 + c = x_3$, $ax_3^2 + bx_3 + c = x_1$, or else $ax_2^2 + bx_2 + c = x_1$, $ax_3^2 + bx_3 + c = x_3$. The latter is eliminated by the remark that x_3 cannot satisfy a second degree equation with rational coefficients. Thus we have the following:

$$x_2 = ax_1^2 + bx_1 + c;$$
$$x_3 = ax_2^2 + bx_2 + c;$$
$$x_1 = ax_3^2 + bx_3 + c.$$

Therefore

$$\sqrt{D}=(x_2-x_1)(x_3-x_2)(x_1-x_3)=$$
$$=[ax_1^2+(b-1)x_1+c][ax_2^2+(b-1)x_2+c][ax_3^2+(b-1)x_3+c]$$

is a rational number since it is a symmetric function of the roots x_1, x_2, x_3 with rational coefficients. This shows the necessity of the given condition.

Now suppose that the discriminant D is the square of a rational number d. Then

$$x_2-x_3=\frac{d}{(x_1-x_2)(x_1-x_3)}=\frac{d}{3x_1^2+2ax_1+b}.$$

On the other hand,

$$x_2 + x_3 = -a - x_1.$$

This shows that x_2, x_3 are rational functions of x_1. The condition has been proved necessary.

861 a) $\frac{2+\sqrt{2}+\sqrt{6}}{4}$; b) $\frac{-3+7\sqrt[3]{2}-\sqrt[3]{4}}{23}$;

c) $1+3\sqrt[4]{2}+2\sqrt{2}-\sqrt[4]{8}$.

862 a) $\frac{\alpha^2-\alpha+1}{3}$; b) $17\alpha^2-3\alpha+55$; c) $3-10\alpha+8\alpha^2-3\alpha^3$;

d) the denominator is 0 for α equal to one of the roots in the equation.

863 $$mx_1^2+nx_1+p=\frac{(pm-bm^2+amn-n^2)x_1+(amp-np-cm^2)}{mx_1+ma-n}.$$

864 If $x_2=\frac{\alpha x_1+\beta}{\gamma x_1+\delta}$, then $x_3=\frac{\alpha x_2+\beta}{\gamma x_2+\delta}$

and

$$x_1=\frac{\alpha x_3+\beta}{\gamma x_3+\delta}=\frac{(\alpha^2+\beta\gamma)x_2+(\alpha+\delta)\beta}{(\alpha+\delta)\gamma x_2+(\beta\gamma+\delta^2)}.$$

On the other hand $x_1 = \dfrac{-\delta x_2 + \beta}{\gamma x_2 - \alpha}$, which shows the necessity of the relation

$$\alpha\delta - \beta\gamma = (\alpha + \delta)^2.$$

865 Suppose

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = a_0 (x - x_1)(x - x_2) \ldots (x - x_n).$$

Then

$$a_0 x^n - a_1 x^{n-1} + \ldots + (-1)^n a_n = a_0 (x + x_1)(x + x_2) \ldots (x + x_n).$$

Multiplying these equations we obtain

$$a_0^2 (x^2 - x_1^2)(x^2 - x_2^2) \ldots (x^2 - x_n^2)$$
$$= (a_0 x^n + a_2 x^{n-2} + \ldots)^2 - (a_1 x^{n-1} + a_3 x^{n-3} + \ldots)^2.$$

Thus we see that, to carry out the transformation $y = x^2$, we must replace x^2 by y in the equation

$$(a_0 x^n + a_2 x^{n-2} + \ldots)^2 - (a_1 x^{n-1} + a_3 x^{n-3} + \ldots)^2 = 0.$$

866 Replace x^3 by y in the equation

$$(a_0 x^n + a_3 x^{n-3} + \ldots)^3 + (a_1 x^{n-1} + a_4 x^{n-4} + \ldots)^3$$
$$+ (a_2 x^{n-2} + a_5 x^{n-5} + \ldots)^3 - 3 (a_0 x^n + a_3 x^{n-3} + \ldots)$$
$$\times (a_1 x^{n-1} + a_4 x^{n-4} + \ldots)(a_2 x^{n-2} + a_5 x^{n-5} + \ldots) = 0.$$

867 If a polynomial $x^n + a_1 x^{n-1} + \ldots$ has integral coefficients, and if the moduli of all its roots are bounded by 1, then the coefficients must be bounded:

$$|a_k| \leqslant \frac{n(n-1)\ldots(n-k+1)}{k!}.$$

Therefore there is no more than a finite number of polynomials of degree n with this property.

Let $f = x^n + a_1 x^{n-1} + \ldots + a_n,\ a_n \neq 0$ be one such polynomial; let $x_1, x_2, \cdots, x_n$ be its

roots. Set $f_m = (x - x_1^m)(x - x_2^m)\cdots(x - x_n^m)$. All the polynomials f_m so constructed will have integral coefficients and roots not exceeding 1 in modulus. Therefore the various polynomials f_m, $m = 1, 2, \cdots$ must comprise only a finite number of different ones. Thus we can choose an infinite sequence $m_0 < m_1 < m_2 < \ldots$, such that $f_{m_0} = f_{m_1} = f_{m_2} = \cdots$. That is,

$$
\begin{aligned}
x_1^{m_i} &= x_{\alpha_1}^{m_0}, \\
x_2^{m_i} &= x_{\alpha_2}^{m_0}, \\
&\cdots\cdots \\
x_n^{m_i} &= x_{\alpha_n}^{m_0},
\end{aligned}
$$

where $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a rearrangement of the integers $1, 2, \cdots, n$. As we have an infinite number of such relations, and as there is only a finite number of permutations on n symbols, we can find two (in fact an infinite number) of positive exponents m_{i_1} , m_{i_2} , that corresponds to the same permutation. Thus we can make the following assertion:

$$
\begin{aligned}
x_1^{m_{i_1}} &= x_1^{m_{i_2}}; \\
x_2^{m_{i_1}} &= x_2^{m_{i_2}}; \\
&\cdots\cdots \\
x_n^{m_{i_1}} &= x_n^{m_{i_2}}.
\end{aligned}
$$

Since $a_n \neq 0$, the numbers $x_1, x_2, \cdots, x_n$ are nonzero and are therefore all roots of unity; in fact correspond to the same exponent $m_{i_2} - m_{i_1}$.

868 Let $F(x_1, x_2, \cdots, x_n)$ be an alternating polynomial, that is, one that changes to its negative under an odd permutation of the variables. Since

$$F(x_2, x_2, x_3, \ldots, x_n) = -F(x_2, x_2, \ldots, x_n) = 0,$$

$F(x_1, x_2, \cdots, x_n)$ must be divisible by $x_1 - x_2$ and in general must be divisible by the product $\Delta = \prod_{i>k}(x_i - x_k)$. Since Δ is also an alternating polynomial, the polynomial F/Δ must be a symmetric polynomial.

869 Let the given polynomial be φ. We know that φ is invariant under an even permutation; the group of all permutations can change φ at most into two forms: even permutations leave it invariant; odd permutations change it to $\overline{\varphi}$. Moreover, every odd permutation must change $\overline{\varphi}$ to φ. Now we note that $\varphi + \overline{\varphi}$ is invariant under the entire permutation group; $\varphi - \overline{\varphi}$ is changed to its negative by an every odd permutation. Therefore

$$\varphi = \frac{\varphi + \overline{\varphi}}{2} + \frac{\varphi - \overline{\varphi}}{2} = F_1 + F_2\Delta.$$

See problem 868.

870 $(f_1^2 - f_2)\Delta$, where f_1, f_2 are elementary symmetric functions in $x_1, x_2, \cdots, x_n$

871 $u^3 + a(\alpha+\beta+\gamma)u^2 +$
$[(\alpha^2+\beta^2+\gamma^2)b + (\alpha\beta+\alpha\gamma+\beta\gamma)\times(a^2-b)]u + c(\alpha^3+\beta^3+\gamma^3) +$
$\frac{ab-3c}{2}(\alpha^2\beta+\alpha\beta^2+\alpha^2\gamma+\alpha\gamma^2$
$+\beta^2\gamma+\beta\gamma^2) + \alpha\beta\gamma(a^3-3ab+6c) + \frac{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}{2}\sqrt{\Delta} = 0,$

where Δ is the discriminant of the given equation.

872 $u^3 - 3pp'u - \frac{27qq' + \sqrt{\Delta\Delta'}}{2} = 0$, where Δ, Δ' are the discriminants of the given equations.

873 Let $y = ax^2 + bx + c$ be the Tschirnhaus transformation connecting the given equations. Then their roots can be numbered so that

$$x_1y_1 + x_2y_2 + x_3y_3 = a\left(x_1^3 + x_2^3 + x_3^3\right) + b\left(x_1^2 + x_2^2 + x_3^2\right) + c\left(x_1 + x_2 + x_3\right)$$

is a rational number. Therefore one of the equations

$$u^3 - 3pp'u - \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2} = 0$$

must have rational roots (see problem 872). Therefore $\sqrt{\Delta\Delta'}$ must be a rational number. The condition is seen to be necessary.

Conversely, suppose that the equation

$$u^3 - 3pp'u - \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2} = 0 \qquad (*)$$

has a rational root u.

We compute the discriminant of the equation (*) and find $\frac{27^2}{4}\left(q\sqrt{\Delta'} - \sqrt{\Delta}q'\right)^2$. Since the ratio Δ'/Δ is the square of a rational number by hypothesis, the discriminant is Δ times the square of a rational number. Now consider the other two roots, u', u''. The product $u'u''$ is a rational number; so that the difference $u' - u''$ can be expressed rationally in terms of the discriminant and u.

Thus u' - u'' is a rational number times $\sqrt{\Delta}$.

The following system of equations is satisfied by y_1, y_2, y_3:

$$y_1+y_2+y_3=0,$$
$$x_1y_1+x_2y_2+x_3y_3=u,$$
$$(x_2-x_3)\,y_1+(x_3-x_1)\,y_2+(x_1-x_2)\,y_3=u'-u''=r\sqrt{\Delta}.$$

From these equations we can obtain by elimination:

$$y_1=\frac{-3ux_1+(x_2-x_3)\,r\sqrt{\Delta}}{6p}.$$

But $(x_2 - x_3)\sqrt{\Delta}$ can be expressed rationally in terms of x_1. The condition is seen to be sufficient.

874 The variables $x_1, x_2, \cdots, x_n$ can be written as linear combinations in f_1, $\eta_1, \eta_2, \ldots, \eta_{n-1}$. Therefore, any polynomial in $x_1, x_2, \cdots, x_n$ can be written as a polynomial in f_1, $\eta_1, \eta_2, \ldots, \eta_{n-1}$:

$$F(x_1, x_2, \ldots, x_n)=\sum Af_1^{\alpha_0}\eta_1^{\alpha_1}\eta_2^{\alpha_2}\ldots\eta_{n-1}^{\alpha_{n-1}}.$$

A circular permutation of the variables $x_1, x_2, \cdots, x_n$ adds the factor $\varepsilon^{-(\alpha_1+2\alpha_2+\ldots+(n-1)\alpha_{n-1})}$ to a typical term $Af_1^{\alpha_0}\eta_1^{\alpha_1}\eta_2^{\alpha_2}\ldots\eta_{n-1}^{\alpha_{n-1}}$. Therefore a necessary condition that $F(x_1, x_2, \cdots, x_n)$ be invariant under all circular permutations is that $\alpha_1+2\alpha_2+\ldots+(n-1)\alpha_{n-1}$ be divisible by n.

875 A possible choice is

$$f_1,\ \eta_1^n,\ \eta_2\eta_1^{-2},\ \ldots,\ \eta_{n-1}\eta_1^{-(n-1)}.$$

876 Set $\eta_1 = x_1 + x_2\varepsilon + x_3\varepsilon^2$; $\eta_2 = x_1 + x_2\varepsilon^2 + x_3\varepsilon$, where $\varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then $\frac{\eta_1^2}{\eta_2} = \varphi_1 + i\sqrt{3}\varphi_2$, where φ_1, φ_2 are certain rational functions in x_1, x_2, x_3 with rational coefficients, invariant under a circular permutation of x_1, x_2, x_3. It remains to show that every rational function of x_1, x_2, x_3 invariant under circular permutations can be expressed in terms of $f_1 = x_1 + x_2 + x_3$, φ_1, φ_2. Indeed it is enough to show this for $\eta_2\eta_1^{-2}$, η_1^3. But

$$\eta_2\eta_1^{-2} = \frac{1}{\varphi_1 + i\varphi_2\sqrt{3}};$$
$$\eta_1^3 = \left(\frac{\eta_1^2}{\eta_2}\right)^2 \cdot \frac{\eta_2^2}{\eta_1} = (\varphi_1 + i\varphi_2\sqrt{3})^2(\varphi_1 - i\varphi_2\sqrt{3}).$$

877 For n = 4,

$$\eta_1 = x_1 + ix_2 - x_3 - ix_4,$$
$$\eta_2 = x_1 - x_2 + x_3 - x_4,$$
$$\eta_3 = x_1 - ix_2 - x_3 + ix_4.$$

Set $\theta_1 = \eta_1\eta_3$; $\theta_2 + i\theta_3 = \frac{\eta_1\eta_2}{\eta_3}$; $\theta_2 - i\theta_3 = \frac{\eta_3\eta_2}{\eta_1}$. Now θ_1, θ_2, θ_3 are rational functions of x_1, x_2, x_3, x_4 with rational coefficients; they are invariant under circular permutations. As in problem 876 one sees that the function $f = x_1 + x_2 + x_3 + x_4$ together with θ_1, θ_2, θ_3 form a basis for the functions needed. Indeed,

$$\eta_2\eta_1^{-2} = \frac{\theta_2 - i\theta_3}{\theta_1}; \quad \eta_3\eta_1^{-3} = \frac{\theta_2 - i\theta_3}{\theta_1(\theta_2 + i\theta_3)}; \quad \eta_1^4 = \frac{\theta_1^4(\theta_2 + i\theta_3)}{\theta_2 - i\theta_3}.$$

878 Set

$$\begin{aligned}
\eta_1 &= x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + x_4\varepsilon^4 + x_5,\\
\eta_2 &= x_1\varepsilon^2 + x_2\varepsilon^4 + x_3\varepsilon + x_4\varepsilon^3 + x_5,\\
\eta_3 &= x_1\varepsilon^3 + x_2\varepsilon + x_3\varepsilon^4 + x_4\varepsilon^2 + x_5,\\
\eta_4 &= x_1\varepsilon^4 + x_2\varepsilon^3 + x_3\varepsilon^2 + x_4\varepsilon + x_5.
\end{aligned}$$

Consider the rational function $\lambda_1 = \frac{\eta_1\eta_2}{\eta_3}$. If we replace 1 by $-\varepsilon-\varepsilon^2-\varepsilon^3-\varepsilon^4$, we expand λ_1 in powers of ε:

$$\lambda_1 = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \varepsilon^4\varphi_4.$$

The coefficients φ_1, φ_2, φ_3, φ_4 are rational numbers. We multiply the preceding equation by ε, ε^2, ε^3, ε^4, and obtain

$$\begin{aligned}
\lambda_2 &= \frac{\eta_2\eta_4}{\eta_1} = \varepsilon^2\varphi_1 + \varepsilon^4\varphi_2 + \varepsilon\varphi_3 + \varepsilon^3\varphi_4,\\
\lambda_3 &= \frac{\eta_3\eta_1}{\eta_4} = \varepsilon^3\varphi_1 + \varepsilon\varphi_2 + \varepsilon^4\varphi_3 + \varepsilon^2\varphi_4,\\
\lambda_4 &= \frac{\eta_4\eta_3}{\eta_2} = \varepsilon^4\varphi_1 + \varepsilon^3\varphi_2 + \varepsilon^2\varphi_3 + \varepsilon\varphi_4.
\end{aligned}$$

The functions f, φ_1, φ_2, φ_3, φ_4 form a basis for the functions invariant under circular permutations. This is shown by the following set of four equations:

$$\begin{aligned}
\eta_2\eta_1^{-2} &= \lambda_1^{-1}\lambda_2\lambda_4^{-1},\\
\eta_3\eta_1^{-3} &= \lambda_1^{-2}\lambda_2\lambda_4^{-1},\\
\eta_4\eta_1^{-4} &= \lambda_1^{-2}\lambda_2\lambda_3^{-1}\lambda_4^{-1},\\
\eta_1^{5} &= \lambda_1^{3}\lambda_2^{-1}\lambda_3\lambda_4^{2}.
\end{aligned}$$

CHAPTER VII - SOLUTIONS

LINEAR ALGEBRA

879 a) Dimension $r = 2$; possible basis vectors X_1, X_2;

b) Dimension $r = 2$; possible basis vectors X_1, X_2;

c) Dimension $r = 2$; possible basis vectors X_1, X_2.

880 a) The intersection has dimension 1 with basis vector

$Z = (5, -2, -3, -4) = X_1 - 4X_2 = 3Y_1 - Y_2$.

The sum has dimension 3 with possible basis Z, X_1, Y_1.

b) The sum is the same as the first space; the intersection is the same as the second.

c) The sum is the entire 4-dimensional space; the intersection is the null vector only.

881 a) $\left(\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$; b) $(1, 0, -1, 0)$.

882 a) $x_1' = \frac{x_1 + x_2 - x_3 - x_4}{2}$, $x_2' = \frac{x_1 - x_2 + x_3 - x_4}{2}$,

$x_3' = \frac{x_1 - x_2 - x_3 + x_4}{2}$, $x_4' = \frac{-x_1 + x_2 + x_3 + x_4}{2}$;

b) $x_1' = x_2 - x_3 + x_4$, $x_2' = -x_1 + x_2$, $x_3' = x_4$,

$x_4' = x_1 - x_2 + x_3 - x_4$.

883 $x_1'x_2' + x_3'x_4' = \frac{1}{8}$.

884 Let $a_0 + a_1 \cos x + a_2 \cos^2 x + \ldots + a_n \cos^n x =$

$b_0 + b_1 \cos x + b_2 \cos 2x + \ldots + b_n \cos nx$.

Then $a_0 = b_0 - b_2 + b_4 - \ldots,$

$$a_k = 2^{k-1} [b_k + \sum_{1 \leqslant p \leqslant \frac{n-k}{2}} (-1)^p \frac{(k+2p)(k+p-1)(k+p-2)\ldots(k+1)}{p!} b_{k+2p}.$$

$$b_0 = a_0 + \sum_{1 \leqslant p \leqslant \frac{n}{2}} 2^{-2p} C_p^{2p} a_{2p},$$

$$b_k = 2^{1-k}\left(a_k + \sum_{1 \leqslant p \leqslant \frac{n-k}{2}} 2^{-2p} C_p^{k+2p} a_{k+2p}\right).$$

885 For the first line the intersection point has coordinates (14/3, 1/9, 7/9, 11/9). For the second line the intersection point is (42, 1, 7, 11).

886 The lines $X_0 + tX_1$, $Y_0 + tY_1$ lie in the manifold $X_0 + t(Y_0 - X_0) + t_1X_1 + t_2Y_1$.

887 In n-dimensional space the problem is solvable if and only if the lines $X_0 + tX_1$, $Y_0 + tY_1$ are such that the vectors X_0, Y_0, X_1, Y_1 are linearly dependent. This occurs if and only if the lines lie in a three-dimensional subspace through the origin.

888 The planes $X_0 + t_1X_1 + t_2X_2$, $Y_0 + t_1Y_1 + t_2Y_2$ are subsets of the manifold

$X_0 + t(Y_0 - X_0) + t_1X_1 + t_2X_2 + t_3Y_1 + t_4Y_2$.

889 There are 6 possible cases:

1) The planes have no common point and lie in no single 4-dimensional linear manifold (the planes are fully skewed).

2) The planes have no common point but do lie in a 4-dimensional manifold but in no 3-dimensional manifold. (Skew-parallel planes.)

3) The planes have no common point but do lie in a 3-dimensional manifold (parallel planes).

4) The planes have a single common point. In this case they lie in a 4-dimensional manifold but in no 3-dimensional manifold.

5) The planes intersect in a line (have a common line).

6) The planes coincide.

If n = 3 (in 3-dimensional space) only cases 3, 5, 6 can occur.

890 Let $Q = X_0 + P$ be the linear manifold, where P is the linear space. If $X_1 \in Q$, $X_2 \in Q$, then $X_1 = X_0 + Y_1$, $X_2 = X_0 + X_2$, where Y_1, Y_2 contain P. Then for arbitrary α, the linear combination $\alpha X_1 + (1-\alpha) X_2 = X_0 + \alpha Y_1 + (1-\alpha) Y_2 \in Q$. Conversely, let Q be a collection of vectors containing every linear combination $\alpha X_1 + (1 - \alpha)X_2$ of every two factors X_1, X_2 that it contains. Let X_0 be some fixed vector in Q, let P denote the set of all vectors $Y = X - X_0$. If $Y \in P$, then $cY \in P$ for every c, since $cY = cX + (1 - c)X_0 - X_0$. Moreover if $Y_1 = X_1 - X_0 \in P$, $Y_2 = X_2 - X_0 \in P$, then $\alpha Y_1 + (1-\alpha) Y_2 = \alpha X_1 + (1-\alpha) X_2 - X_0 \in P$ for every α. Now choose some fixed number α, $(\alpha \neq 0, \alpha \neq 1)$ and arbitrary c_1, c_2. Then $(c_1/\alpha)Y_1 \in P$, $c_2/(1 - \alpha) \cdot Y_2 \in P$ for arbitrary Y_1, $Y_2 \in P$. Therefore also

$$c_1 Y_1 + c_2 Y_2 = \alpha \frac{c_1}{\alpha} Y_1 + (1-\alpha) \frac{c_2}{1-\alpha} Y_2 \in P.$$

Therefore P is a linear space, Q is a linear manifold.

Remark. The result is not valid for vector spaces over a field of characteristic 2.

891 a) 9; b) 0.

892 a) 90°; b) 45°; c) $\cos\varphi = \frac{3}{\sqrt{77}}$.

893 $\cos\varphi = \frac{1}{\sqrt{n}}$.

894 $\cos A = \frac{5}{\sqrt{39}}$, $\cos B = \frac{8}{\sqrt{78}}$, $\cos C = -\frac{\sqrt{2}}{3}$.

895 $\sqrt{n}$.

896 If n is odd there are no orthogonal diagonals. If n = 2m, the number of orthogonal diagonals is C_{m-1}^{2m-1}.

897 The points have coordinates given by the rows of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & \dots & 0 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \sqrt{\frac{4}{6}} & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{24}} & \dots & \frac{1}{\sqrt{2n(n-1)}} & \sqrt{\frac{n+1}{2n}} \end{pmatrix}.$$

898 $R=\sqrt{\dfrac{n}{2(n+1)}}$.

Coordinates of the center:

$$\left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \ldots, \frac{1}{\sqrt{2n(n-1)}}, \frac{1}{\sqrt{2(n+1)n}}\right).$$

899 $\left(\dfrac{3}{\sqrt{15}}, \dfrac{1}{\sqrt{15}}, \dfrac{2}{\sqrt{15}}, \dfrac{1}{\sqrt{15}}\right)$.

900 $\left(0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0\right)$.

901 Possible choice of the two vectors is

$$\frac{1}{\sqrt{26}}(0, -4, 3, 1), \quad \frac{1}{3\sqrt{26}}(-13, 5, 6, 2).$$

902 (1, 2, 1, 3), (10, -1, 1, -3), (19, -87, -61, 72).

903 For example, $\begin{pmatrix} 0, & 7, & 3, & -4 & -2 \\ 39, & -37, & 51, & -29 & 5 \end{pmatrix}$.

904 The coefficients of the various equations determine m vectors. The system can be solved by finding a vector (or all the vectors) that are orthogonal to each of the given vectors. The given vectors span a subspace; the solutions fill out a space orthogonal to this subspace (its orthogonal complement). A basis for the solution set is a basis for this orthogonal complement.

905 For example, $\dfrac{1}{\sqrt{6}}(1, 0, 2, -1)$, $\dfrac{1}{\sqrt{498}}(1, 12, 8, 17)$.

906 a) $X' = (3, 1, -1, -2) \in P$, $X'' = (2, 1, -1, 4) \perp P$; b) $X' = (1, 7, 3, 3) \in P$, $X'' = (-4, -2, 6, 0) \perp P$.

907 Let $A_1, A_2, \cdots, A_m$ be linearly independent; let P be the space they span. Further set $X = Y + Z$, $Y \in P$, $Z \perp P$.

Set $Y = c_1A_1 + c_2A_2 + \cdots + c_mA_m$. To obtain a system of equations for $c_1, c_2, \cdots, c_m$, form the scalar product of this last equation with A_i, $i = 1, 2, \cdots, m$, denoting the scalar product by $YA_i = XA_i$. We obtain

$$\begin{aligned}
c_1A_1^2 + c_2A_1A_2 + \ldots + c_mA_1A_m &= XA_1,\\
c_1A_2A_1 + c_2A_2^2 + \ldots + c_mA_2A_m &= XA_2,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\\
c_1A_mA_1 + c_2A_mA_2 + \ldots + c_mA_m^2 &= XA_m.
\end{aligned}$$

Since $A_1, A_2, \cdots, A_m$ are linearly independent, the determinant Δ of this system is not 0.

Solving for $c_1, c_2, \cdots, c_m$, we obtain the following formula for Y:

$$Y = \frac{1}{\Delta} \det \begin{bmatrix} 0 & -A_1 & -A_2 & \ldots & -A_m \\ XA_1 & A_1^2 & A_1A_2 & \ldots & A_1A_m \\ XA_2 & A_2A_1 & A_2^2 & \ldots & A_2A_m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ XA_m & A_mA_1 & A_mA_2 & \ldots & A_m^2 \end{bmatrix}$$

and

$$Z = \frac{1}{\Delta} \det \begin{bmatrix} X & A_1 & A_2 & \ldots & A_m \\ A_1X & A_1^2 & A_1A_2 & \ldots & A_1A_m \\ A_2X & A_2A_1 & A_2^2 & \ldots & A_2A_m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_mX & A_mA_1 & A_mA_2 & \ldots & A_m^2 \end{bmatrix}.$$

These equations are to be interpreted as follows. Each vector is given as a linear combination of the vectors in the first row of the determinant, the coefficients in this linear combination being the respective algebraic co-factors.

The following relations conclude the exercise:

$$Y^2 = Y(X - Z) = YX = \frac{1}{\Delta}\det\begin{bmatrix} 0 & -XA_1 & -XA_2 & \dots & -XA_m \\ A_1X & A_1^2 & A_1A_2 & \dots & A_1A_m \\ A_2X & A_2A_1 & A_2^2 & \dots & A_2A_m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_mX & A_mA_1 & A_mA_2 & \dots & A_m^2 \end{bmatrix}$$

and

$$Z^2 = Z(X - Y) = ZX = \frac{1}{\Delta}\det\begin{bmatrix} X^2 & XA_1 & XA_2 & \dots & XA_m \\ A_1X & A_1^2 & A_1A_2 & \dots & A_1X_m \\ A_2X & A_2A_1 & A_2^2 & \dots & A_2A_m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_mX & A_mA_1 & A_mA_2 & \dots & A_m^2 \end{bmatrix}.$$

908 Let Y be some vector in the space P, and let X' be the orthogonal projection of X onto P. Then

$$\cos(X, Y) = \frac{XY}{|X|\cdot|Y|} = \frac{X'Y}{|X|\cdot|Y|} = \frac{|X'|\cdot|Y|\cdot\cos(X', Y)}{|X|\cdot|Y|}$$
$$= \frac{|X'|}{|X|}\cos(X', Y),$$

where $|X|$ means the length of the vector X. This

formula shows that cos(X, Y) is a maximum when cos(X', Y) = 1, i.e. $Y = \alpha X'$, $\alpha > 0$.

909 a) 45°; b) 90°.

910 $\sqrt{\frac{m}{n}}$.

911 $|X-Y|^2=|(X-X')+(X'-Y)|^2=|X-X'|^2+|X'-Y|^2$

$\geqslant |X-X'|^2$, so that the equality can occur only for Y = X'.

912 a) $\sqrt{7}$; b) $\sqrt{\frac{2}{3}}$.

913 $\frac{n!}{(n+1)(n+2)\dots 2n\sqrt{2n+1}}$.

914 The shortest distance from $X_0 + P$ to $Y_0 + Q$ is equal to the shortest distance from the point $X_0 - Y_0$ to the space P + Q.

915 Place the axis so that one of the vertices lies at the origin and suppose $X_1, X_2, \cdots, X_n$ are vectors from the origin to the remaining vertices. It is easy to see that $X_i^2 = 1$, $X_iX_j = \frac{1}{2}$. The manifold spanned by the first m + 1 vertices is the space $t_1X_1 + \cdots + t_mX_m$. The manifold spanned by the remaining n - m vertices is

$X_n+t_{m+1}(X_{m+1}-X_n)+\dots+t_{n-1}(X_{n-1}-X_n)$. The shortest distance required is the distance from X_n to the space P, spanned by the vectors $X_1, X_2, \dots, X_m, X_n-X_{m+1}, \dots, X_n-X_{n-1}$.

Set $X_n =$

$$t_1X_1 + \dots + t_mX_m\, t_{m+1}(X_n - X_{m+1}) + \dots + t_{n-1}(X_n - X_{n-1}) + Y,$$

where $Y \perp P$. We can obtain simultaneous equations to solve for $t_1, \cdots, t_{n-1}$ by forming the inner product of X_n with X_n, $X_1, \cdots, X_m$, $X_n - X_{m+1}, \cdots,$ $X_n - X_{n-1}$:

$$t_1 + \tfrac{1}{2}t_2 + \dots + \tfrac{1}{2}t_m = \tfrac{1}{2}, \quad t_{m+1} + \tfrac{1}{2}t_{m+2} + \dots + \tfrac{1}{2}t_{n-1} = \tfrac{1}{2},$$
$$\tfrac{1}{2}t_1 + \ \ t_2 + \dots + \tfrac{1}{2}t_m = \tfrac{1}{2}, \quad \tfrac{1}{2}t_{m+1} + \ \ t_{m+2} + \dots + \tfrac{1}{2}t_{n-1} = \tfrac{1}{2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\tfrac{1}{2}t_1 + \tfrac{1}{2}t_2 + \dots + \ \ t_m = \tfrac{1}{2}, \quad \tfrac{1}{2}t_{m+1} + \tfrac{1}{2}t_{m+2} + \dots + \ \ t_{n-1} = \tfrac{1}{2},$$

Thus $t_1 = t_2 = \dots = t_m = \dfrac{1}{m+1},\ t_{m+1} = t_{m+2} = \dots = t_{n-1} = \dfrac{1}{n-m}.$

Therefore $Y = \dfrac{X_{m+1} + X_{m+2} + \dots + X_n}{n-m} - \dfrac{X_1 + X_2 + \dots + X_m}{m+1}.$

Thus the vector that is the common perpendicular to the spaces in question runs between the centroids of their boundary points. The shortest distance itself is given by the length of this vector:

$$|Y| = \sqrt{\frac{(n+1)}{2(n-m)(m+1)}}.$$

916 a) The projection of the vector $(t_1 + 2t_2,\ t_1 - 2t_2,\ t_1 + 5t_2,\ t_1 + 2t_2)$ on the first plane is $(t_1 + 2t_2,\ t_1 - 2t_2,\ 0,\ 0)$. Thus

$$\cos^2\varphi = \frac{2t_1^2 + 8t_2^2}{4t_1^2 + 14t_1t_2 + 37t_2^2} = \frac{2\lambda^2 + 8}{4\lambda^2 + 14\lambda + 37}, \text{ where } \lambda = \frac{t_1}{t_2}.$$

The value of this fraction is a maximum, namely 8/9, for $\lambda = -4$.

b) The angle between an arbitrary vector of the second space and its orthogonal projection on the first space is constant and equal to $\pi/4$.

917 A cube is the set of points the coordinates of which satisfy the inequalities $-\frac{a}{2} \leqslant x_i \leqslant \frac{a}{2}$, $i = 1, 2, 3, 4$. Here a is the length of an edge. Introduce new axes with basis

$$e'_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad e'_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right),$$

$$e'_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \quad e'_4 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right).$$

These vectors are orthogonal; they all have the same length; and their directions coincide with the directions of the diagonals of the cube. In terms of the new axes, the points interior to the cube satisfy the inequalities

$$-a \leqslant x'_1 + x'_2 + x'_3 + x'_4 \leqslant a, \quad -a \leqslant x'_1 + x'_2 - x'_3 - x'_4 \leqslant a,$$
$$-a \leqslant x'_1 - x'_2 + x'_3 - x'_4 \leqslant a, \quad -a \leqslant x'_1 - x'_2 - x'_3 + x'_4 \leqslant a.$$

We can obtain the intersection asked for by setting $x'_1 = 0$. It is the solid that lies in the space spanned by e'_2, e'_3, e'_4; the coordinates of the points in this solid satisfy the inequalities $\pm x'_2 \pm x'_3 \pm x'_4 \leq a$.

This is a regular octohedron bounded by planes that have intercept a on the coordinate axes.

918 $$V^2[B_1, B_2, \ldots, B_m] = \det \begin{bmatrix} B_1^2 & B_1B_2 & \ldots & B_1B_m \\ B_2B_1 & B_2^2 & \ldots & B_2B_m \\ \cdot & \cdot & \cdot & \cdot \\ B_mB_1 & B_mB_2 & \ldots & B_m \end{bmatrix} = \det[B'B] = \{\det B\}^2.$$

It is easy to establish this formula by induction, using the result of problem 907. The nature of the formula shows that the volume has a value independent of the way the vertices are numbered. It is also clear that
$V[cB_1, B_2, \cdots, B_m] = |c| \; V[B_1, B_2, \cdots, B_m]$.

Now let $B_1 = B_1' + B_1''$; let the orthogonal projections of B_1, B_1', B_1'' on the orthogonal complement of the space $(B_2, \cdots, B_m)$ be C_1, C_1', C_1''. It is clear that $C_1 = C_1' + C_1''$. By definition,
$V[B_1, B_2, \ldots, B_m] = |C_1| \cdot V[B_2, \ldots, B_m]$, $V[B_1', B_2, \ldots, B_m] = |C_1'| \cdot V[B_2, \ldots, B_m]$, $V[B_1'', B_2, \ldots, B_m] = |C_1''| \cdot V[B_2, \ldots, B_m]$.
Since $|C_1| \leqslant |C_1'| + |C_1''|$, it follows that
$V[B_1, B_2, \ldots, B_m] \leqslant V[B_1', B_2, \ldots, B_m] + V[B_1'', B_2, \ldots, B_m]$.
Indeed equality is only possible if C_1', C_1'' are collinear and point in the same direction. But this can happen if and only if B_1', B_1'' lie in the space spanned by $B_1, B_2, \cdots, B_m$, and even then only if in the formulas for expressing B_1', B_1'' in terms of $B_1, B_2, \cdots, B_m$ the two coefficients of B_1 have the same sign. This last requirement is paraphrased geometrically by asserting that B_1', B_1'' lie on the same side of the space $(B_2, \cdots, B_m)$ in the space $(B_1, B_2, \cdots, B_m)$.

919 $$V^2[B_1, B_2, \ldots, B_n] = \det \begin{bmatrix} B_1^2 & B_1B_2 & \ldots & B_1B_n \\ B_2B_1 & B_2^2 & \ldots & B_2B_n \\ \cdot & \cdot & \cdot & \cdot \\ B_nB_1 & B_nB_2 & \ldots & B_n^2 \end{bmatrix} = \det[B'B] = \{\det B\}^2,$$

where B is the matrix, the rows of which are the coordinates of the vectors $B_1, B_2, \cdots, B_n$.

920 The following two properties of "volume" follow directly from the definition:

d) $V[B_1+X, B_2, \ldots, B_m] = V[B_1, B_2, \ldots, B_m]$

where X is an arbitrary vector lying in the space $(B_2, \cdots, B_m)$, and the points B_1, $B_1 + X$ are at the same distance from $(B_2, \cdots, B_m)$.

e) $V[B_1, B_2, \ldots, B_m] \leqslant |B_1| \cdot V[B_2, \ldots, B_m]$.

This follows from the fact that the "altitude," that is, the length of the component of the vector B_1 that is orthogonal to $(B_2, \cdots, B_m)$ is no greater than the length of the vector B_1 itself.

Now let $C_1, C_2, \cdots, C_m$ be the orthogonal projections of the vector $B_1, B_2, \cdots, B_m$ onto a space P. Suppose as an induction hypothesis that the relation $V[C_2, \ldots, C_m] \leqslant V[B_2, \ldots, B_m]$ has already been proved. Let B_1' denote the component of the vector B_1 that is orthogonal to $(B_2, \cdots, B_m)$. Let C_1' be the projection of B_1 on P. Since $B_1' - B_1 \in (B_2, \ldots, B_m)$, we conclude that $C_1' - C \in (C_2, \ldots, C_m)$. Therefore

$V[C_1, C_2, \cdots, C_m] = V[C_1', C_2, \ldots, C_m] \leqslant |C_1'| \cdot V[C_2, \ldots, C_m]$.
Since the relation $|C_1'| \leqslant |B_1'|$ is true, the induction hypothesis $V[C_2, \ldots, C_m] \leqslant V[B_2, \ldots, B_m]$ gives the result

$V[C_1, C_2, \ldots, C_m] \leqslant |B_1'| \cdot V[B_2, \ldots, B_m] = V[B_1, B_2, \ldots, B_m]$.

To complete the proof one need merely remark the truth of the result in one dimension.

921 If every vector A_i is orthogonal to every vector B_j, the formula

$V[A_1, \ldots, A_m, B_1, \ldots, B_k] = V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k]$

follows from the formula for the square of a general volume. To establish the theorem in general we define $C_1, \cdots, C_k$ as the projections of $B_1, \cdots, B_k$ on the orthogonal complement of $(A_1, \cdots, A_m)$. The preceding problem gives the result $V[C_1, \ldots, C_k] \leqslant V[B_1, \ldots, B_k]$, and the following chain of relations establishes the general result.

$$V[A_1, \ldots, A_m, B_1, \ldots, B_k] = V[A_1, \ldots, A_m, C_1, \ldots, C_k] =$$
$$= V[A_1, \ldots, A_m] \cdot V[C_1, \ldots, C_k] \leqslant V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k].$$

See problem 518, which is essentially equivalent to this one.

922 As shown in 920, $V[A_1, \ldots, A_m] \leqslant |A_1| \cdot V[A_2, \ldots, A_m]$. The result follows directly from this formula.

See problem 519, which is essentially identical with this one.

923 Similar solids in n-dimensional space have volumes proportional to the n-th power of the coefficient of similarity. This is clear for parallelepipeds, and the volume of a general solid can be thought of as the limit of the sum of volumes of parallelepipeds. Therefore the volume $V_n(R)$ of an n-dimensional sphere of radius R is $V_n(1)R^n$.

The volume of $V_n(1)$ can be computed by the method of Archimedes. We cut the sphere by (n - 1)-dimensional hyperplanes and use Cavalieri's principle.

Let x be the distance of a typical hyperplane from the center. The trace of this hyperplane is a (n - 1)-dimensional sphere of radius $\sqrt{1-x^2}$. Therefore

$$V_n(1)=2\int_0^1 V_{n-1}(\sqrt{1-x^2})\,dx=2V_{n-1}(1)\int_0^1(1-x^2)^{\frac{n-1}{2}}\,dx$$

$$=V_{n-1}(1)\int_0^1 t^{\frac{n-1}{2}}(1-t)^{-\frac{1}{2}}\,dt=V_{n-1}(1)\,B\left(\frac{n+1}{2},\ \frac{1}{2}\right)$$

$$=V_{n-1}(1)\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}.$$

From this, it follows that $V_n(1)=\dfrac{n^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$.

924 The polynomials $1, x, \cdots, x^n$ form a basis. The square of the volume of the corresponding parallelepipeds is

$$\det\begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{bmatrix} = \frac{[1!\,2!\ldots n!]^3}{(n+1)!\,(n+2)!\ldots(2n+1)!}.$$

925 a) $\lambda_1 = 1,\ X_1 = c(1,\ -1);\ \lambda_2 = 3,\ X_2 = c(1,\ 1);$

b) $\lambda_1 = 7,\ X_1 = c(1,\ 1);\ \lambda_2 = -2,\ X_2 = c(4,\ -5);$

c) $\lambda_1 = ai,\ X_1 = c(1,\ i);\ \lambda_2 = -ai,\ X_2 = c(1,\ -i);$

d) $\lambda_1 = 2,\ X_1 = c_1(1,\ 1,\ 0,\ 0) + c_2(1,\ 0,\ 1,\ 0) + c_3(1,\ 0,\ 0,\ 1);$
$\lambda_2 = -2,\ X_2 = c(1,\ -1,\ -1,\ -1);$

e) $\lambda = 2,\ X = c_1(-2,\ 1,\ 0) + c_2(1,\ 0,\ 1);$

f) $\lambda = -1,\ X = c(1,\ 1,\ 1,\ -1);$

g) $\lambda_1 = 1, X_1 = c_1(1, 0, 1) + c_2(0, 1, 0,); \lambda_2 = -1, X_2 = c(1, 0, -1);$

h) $\lambda_1 = 0,\ X_1 = c(3,\ -1,\ 2);\ \lambda_{2,3} = \pm\sqrt{-14},$
$X_{2,3} = c\left(3 \pm 2\sqrt{-14},\ 13,\ 2 \mp 3\sqrt{-14}\right);$

i) $\lambda_1 = 1,\ X_1 = c(3,\ -6,\ 20);\ \lambda_2 = -2,\ X_2 = c(0,\ 0,\ 1);$

j) $\lambda_1 = 1,\ X_1 = c(1,\ 1,\ 1);\ \lambda_2 = \varepsilon,\ X_2 = c(3 + 2\varepsilon,\ 2 + 3\varepsilon,\ 3 + 3\varepsilon);$
$\lambda_3 = \varepsilon^2,\ X_3 = c(3 + 2\varepsilon^2,\ 2 + 3\varepsilon^2,\ 3 + 3\varepsilon^2),$

where $\varepsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}.$

926 The proper values of the matrix A^{-1} are the reciprocals of the proper values of the matrix A. For if $\det[A^{-1} - \lambda E] = 0$, then $\det[E - \lambda A] = 0$, $\det[A - \lambda^{-1}E] = 0$.

927 The proper values of the matrix A^2 are the squares of the proper values of the matrix A. Indeed suppose $\det[A - \lambda E] = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$. Then $\det[A + \lambda E] = (\lambda_1 + \lambda)(\lambda_2 + \lambda) \ldots (\lambda_n + \lambda)$. If these two equations are multiplied, and if λ^2 is replaced by μ, we obtain

$$\det[A^2 - \mu E] = (\lambda_1^2 - \mu)(\lambda_2^2 - \mu) \cdots (\lambda_n^2 - \mu).$$

If the proper values of a matrix are defined in the intrinsic way: $AX = X\lambda$, then it follows that $A^2X = AX\lambda = X\lambda^2$. This argument must be slightly modified if the matrix A is deficient, that is if it has fewer than n proper vectors. This modification can consist either of a continuity argument, or a full use of the Jordan normal form.

928 The proper values of the matrix A^m are the m-th powers of the proper values of the matrix A.

The proof is similar to that of problem 927; the relation

$$\det[A - \lambda E] = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

is changed to n - 1 other relations by replacing λ by $\lambda\epsilon, \lambda\epsilon^2, \cdots, \lambda\epsilon^{n-1}$, where ϵ is a primitive n-th root of unity. As in the preceding problem the n equations are then multiplied. Finally λ^n is replaced by μ. The alternative argument of problem 927 can also be used.

929 From the relation $f(A) = b_0(A - \xi_1 E)\dots(A - \xi_m E)$, it follows that $\det[F(A)] = b_0^n \cdot \det[A - \xi_1 E] \cdots \det[A - \xi_m E] = b_0^n F(\xi_1) \cdots F(\xi_m)$.

930 Let $F(\lambda) = |A - \lambda E| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda)$, and $f(x) = b_0(x - \xi_1)(x - \xi_2)\dots(x - \xi_m)$. Then

$$\det[f(A)] = b_0^n \prod_{i=1}^{n}\prod_{k=1}^{m}(\lambda_i - \xi_k) = f(\lambda_1)f(\lambda_2)\dots f(\lambda_n).$$

931 Set $\varphi(x) = f(x) - \lambda$ and apply the result of the preceding problem to obtain:

$$\det[f(A) - \lambda E] = (f(\lambda_1) - \lambda)(f(\lambda_2) - \lambda)\dots(f(\lambda_n) - \lambda).$$

This shows that the proper values of the matrix $f(A)$ are $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)$. The second proof in 927 can be applied if f is any function expansible in a convergent infinite series.

The proper values of a triangular matrix are the diagonal elements. The proper values of any matrix are the same as those of a similar matrix. But by Schur's theorem every matrix is similar to a triangular matrix. A very short argument shows that the diagonal elements of a power of a triagonal matrix are the corresponding powers of the diagonal elements of the original matrix. Thus we have another proof of the results of problems 927, 928, 929, 930, and 931.

Schur's theorem is easily established by induction; the induction hypothesis is that the theorem is valid for every matrix of dimension $n - 1$. To complete the proof it is only necessary

to know that every n-th order matrix has at least one proper value. Let λ be a proper value of the matrix A; let e_1 be the corresponding proper vector. Construct the matrix U, a unitary matrix having e_1 for first row (the possibility of constructing such a matrix is known as the Gram-Schmidt orthogonalization process). Then U transforms A into block triangular form:

$$U^*AU = \begin{bmatrix} \lambda & a \\ 0 & A_1 \end{bmatrix} .$$

By the induction hypothesis, there is a unitary matrix

$$\operatorname{diag}\{1, V\}$$

that transforms U^*AU into triagonal form. Therefore $U \cdot \operatorname{diag}\{1, V\}$ transforms A into triangular form, as was to be proved.

932 Let X be a proper vector of the matrix A, and let the corresponding proper value be λ. Then

$$\begin{aligned} EX &= X, \\ AX &= X\lambda, \\ A^2X &= X\lambda^2 \\ &\cdots\cdots \\ A^mX &= X\lambda^m \end{aligned}$$

If the vector equations in this array are multiplied by arbitrary scalars, and added, we obtain the following relation for an arbitrary polynomial f:

$$f(A)\, X = X\, f(\lambda).$$

Thus X is a proper vector of f(A) and the corresponding proper value is $f(\lambda)$.

If A is not deficient, that is if A has n linearly independent proper vectors, then, taking f as the characteristic polynomial of A, we see that $f(A)X = 0$ for every proper vector. But this relation must then be true for every vector in the space, since the proper vectors are linearly independent, and therefore span the space. In this case therefore, the polynomial f(A) must be the null matrix, since this is the only matrix that transforms the entire space into the null space. Thus in this case it has been proved that the matrix A satisfies its characteristic equation.

933 The proper values of A^2 are n, -n, with multiplicity $(n + 1)/2$, $(n - 1)/2$. Thus the proper values of A are $+\sqrt{n}$, $-\sqrt{n}$, $+i\sqrt{n}$, $-i\sqrt{n}$. If the multiplicities of these roots are a, b, c, d, we have $a + b = (n + 1)/2$, $c + d = (n - 1)/2$. The sum of the proper values of a matrix is the sum of the elements in the principal diagonal. Thus

$$[a-b+(c-d)i]\sqrt{n} = 1+\varepsilon+\varepsilon^4+\dots+\varepsilon^{(n-1)^2}.$$

By problem 126, the absolute value of the right member of this equation is $\sqrt{n}$. Therefore

$$(a - b)^2 + (c - d)^2 = 1.$$

Note that $c - d$, $c + d$ are both even or both odd;

thus

$a - b = 0$, $c - d = \pm 1$, if $(n-1)/2$ is odd

and

$a - b = \pm 1$, $c - d = 0$, if $(n-1)/2$ is even.

Finally we obtain for $n = 1 + 4k$:

$c = d = k$; $a = k + 1$, $b = k$ or $a = k$, $b = k + 1$;

for $n = 3 + 4k$

$a = b = k + 1$; $c = k + 1$, $d = k$ or $c = k$, $d = k + 1$

At the moment, proper values are determined to within sign. To obtain the sign we use the fact that the product of the proper values is the determinant of the matrix. Problem 299 enables us to show that for $n = 1 + 4k$, we have

$$a = k + 1, \ b = k;$$

for $n = 3 + 4k$, we have

$$c = k + 1, \ d = k.$$

This determines the characteristic numbers completely.

934 $1+\varepsilon+\varepsilon^4+\ldots+\varepsilon^{(n-1)^2}=+\sqrt{n}$ for $n = 4k + 1$,

$1+\varepsilon+\varepsilon^4+\ldots+\varepsilon^{(n-1)^2}=+i\sqrt{n}$ for $n = 4k + 3$.

935 Set $\frac{x}{y}=\alpha^n$. Then $\lambda_k=y\frac{\alpha\varepsilon_k-\alpha^n}{1-\alpha\varepsilon_k}$, where $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{a}$, $k=0, 1, \ldots, n-1$.

b) $\lambda_k=a_1+a_2\varepsilon_k+a_2\varepsilon_k^2+\ldots+a_n\varepsilon_k^{n-1}$.

c) $\lambda_k=2i\cos\frac{k\pi}{n+1}$, $k=1, 2, \ldots, n$.

936
$$A \times B - \lambda E_{mn} = \begin{pmatrix} a_{11}B - \lambda E_m & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B - \lambda E_m & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B - \lambda E_m \end{pmatrix},$$

so that by problem 537, we see that

$$\det[A \times B - \lambda E_{mn}] = \det[\varphi(B)],$$

where

$$\varphi(x) = |Ax - \lambda E_n| = \prod_{i=1}^{n} (\alpha_i x - \lambda).$$

By problem 930,

$$\det[\varphi(B)] = \prod_{k=1}^{m} \varphi(\beta_k) = \prod_{i=1}^{n} \prod_{k=1}^{m} (\alpha_i \beta_k - \lambda).$$

Thus the proper values of $A \times B$ are the numbers $\alpha_i\beta_k$, where α_i are the proper values of A, β_k are the proper values of B.

937 If A is a nonsingular matrix, then $\det[BA - \lambda E] = \det[A^{-1}(AB - \lambda E)A] = \det[A^{-1}] \cdot \det[AB - \lambda E] \cdot \det[A] = \det[AB - \lambda E]$.

To improve this result to the case of singular matrices, we can think of it as an identity in the n^2 elements of A, since it has been proved in all but a minority of the cases. Indeed A is singular only when its coefficients satisfy a polynomial equation. A more direct proof consists in noting the relations

$$\det[B(A - \mu I) - \lambda I] = \det[(A - \mu I)B - \lambda I],$$

which is obviously true for all but a finite number of values of μ. Since this is a polynomial

identity, it must be true for all the values of μ, including $\mu = 0$.

938 Complete the matrices A, B to square matrices A', B' of order n by appending n - m rows of zeros to A, and n - m columns of zeros to B. Then BA = B'A', and A'B' can be obtained from AB by adding a border of zeros. The result to be proved can be obtained by applying the preceding problem.

Problems 939, 940, 941 do not have unique answers. We give below the answers that are as close to triangular in appearance as possible.

939 a) $x_1'^2 + x_2'^2 + x_3'^2$,

$$x_1' = x_1 + x_2,\quad x_2' = x_2 + 2x_3,\quad x_3' = x_3;$$

b) $-x_1'^2 + x_2'^2 + x_3'^2$,

$$x_1' = x_1,\quad x_2' = x_1 - 2x_2,\quad x_3' = x_1 + x_3;$$

c) $x_1'^2 - x_2'^2 - x_3'^2$,

$$x_1' = \frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3,\quad x_2' = \frac{1}{2}x_1 - \frac{1}{2}x_2,\quad x_3' = x_3;$$

d) $x_1'^2 + x_2'^2 - x_3'^2$,

$$x_1' = x_1 - x_2 + x_3 - x_4,\quad x_2' = x_2 + x_3 + x_4,\quad x_3' = x_2 - x_3 + 2x_4,\quad x_4' = x_4;$$

e) $x_1'^2 - x_2'^2 + x_3'^2 - x_4'^2$,

$$x_1' = x_1 + \frac{1}{2}x_2,\quad x_2' = \frac{1}{2}x_2,\quad x_3' = \frac{1}{2}x_3 + \frac{1}{2}x_4,\quad x_4' = \frac{1}{2}x_3 - \frac{1}{2}x_4.$$

940 $x_1'^2+\frac{3}{4}x_2'^2+\frac{4}{6}x_3'^2+\ldots+\frac{n+1}{2n}x_n'^2.$

Here we give just the matrix of coefficients for expressing $x_1', x_2', \cdots, x_n'$ in terms of $x_1, x_2, \cdots, x_n$:

$$\begin{pmatrix} 1 & 1/2 & 1/2 & & \ldots & 1/2 \\ 0 & 1 & 1/3 & & \ldots & 1/3 \\ 0 & 0 & 1 & 1/4 & \ldots & 1/4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

941 $$\left(\frac{x_1+x_2}{2}+x_3+x_4+\ldots+x_n\right)^2-\left(\frac{x_1-x_2}{2}\right)^2$$
$$-\left(x_3+\frac{1}{2}x_4+\ldots+\frac{1}{2}x_n\right)^2$$
$$-\frac{3}{4}\left(x_4+\frac{1}{3}x_5+\ldots+\frac{1}{3}x_n\right)^2-\ldots-\frac{n-1}{2(n-2)}x_n^2.$$

942 The matrix of a positive definite quadratic form has the form A'A, where A is the real nonsingular matrix of coefficients needed to transform the form into the sum of squares of new variables.

To show that the minors are positive, use the result of problem 510.

943 Let $$f=a_{11}x_1^2+\ldots+a_{1n}x_1x_n+$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$+a_{n1}x_nx_1+\ldots+a_{nn}x_n^2$$

be a quadratic form. Set
$$f^{(k)}=a_{11}x_1^2+\ldots+a_{1k}x_1x_k+$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$+a_{k1}x_kx_1+\ldots+a_{kk}x_k^2,$$

$\Delta_k=\det\begin{bmatrix} a_{11} & \ldots & a_{1k} \\ \cdot & \cdot & \cdot \\ a_{k1} & \ldots & a_{kk}\end{bmatrix}$; Let r be the rank of the form f.

Set

$$f = \alpha_1 x_1'^2 + \alpha_2 x_2'^2 + \dots + \alpha_n x_n'^2,$$

where

$$\begin{aligned} x_1' &= x_1 + b_{12}x_2 + \dots + b_{1n}x_n, \\ x_2' &= \phantom{x_1 + b_{12}} x_2 + \dots + b_{2n}x_n, \\ &\dots\dots\dots\dots\dots\dots \\ x_n' &= \phantom{x_1 + b_{12}x_2 + \dots + b_{1n}} x_n. \end{aligned}$$

Since the determinant of the triagonal transformation above is 1, it follows that $D_f = \Delta_n = \alpha_1\alpha_2 \dots \alpha_n$. Take $x_{k+1} = \dots = x_n = 0$, to obtain

$$f^{(k)} = \alpha_1 x_1^{(k)^2} + \alpha_2 x_2^{(k)^2} + \dots + \alpha_k x_k^{(k)^2},$$

where

$$\begin{aligned} x_1^{(k)} &= x_1 + b_{12}x_2 + \dots + b_{1k}x_k, \\ x_2^{(k)} &= \phantom{x_1 + b_{12}} x_2 + \dots + b_{2k}x_k, \\ &\dots\dots\dots\dots\dots\dots \\ x_k^{(k)} &= \phantom{x_1 + b_{12}x_2 + \dots + b_{1k}} x_k. \end{aligned}$$

Thus it follows that $\Delta_k = \alpha_1\alpha_2 \dots \alpha_k$ and that a necessary condition that the form be transformable into a sum of squares by a triagonal transformation is that the inequalities

$$\Delta_1 \neq 0,\ \Delta_2 \neq 0,\ \dots,\ \Delta_r \neq 0$$

should hold. It is easy to see that this condition is also sufficient.

Moreover, $\alpha_k = \Delta_k/\Delta_{k-1}$ if $k \leq r$; $\alpha_k = 0$ if $k > r$.

The discriminant of the form

$$\begin{aligned} f_k(x_{k+1}, \dots, x_n) &= f - \alpha_1 x_1'^2 - \dots - \alpha_k x_k'^2 \\ &= \alpha_{k+1} x_{k+1}'^2 + \dots + \alpha_n x_n'^2 \end{aligned}$$

is given by $\alpha_{k+1}\alpha_{k+2}\ldots\alpha_n = \Delta_n/\Delta_k$.

944 In problem 942 Sylvester's condition was shown to be necessary. In problem 943 it was shown to be sufficient.

945 Let ℓ be some linear form in the variables $x_1, x_2, \cdots, x_n$. We transform the form f by a sequence of transformations all of which have unit determinant, and take the form ℓ as the last of the new variables. Finally we transform f into canonical form by performing a triagonal transformation. The form f takes the form

$$f=\alpha_1 x_1'^2+\alpha_2 x_2'^2+\ldots+\alpha_n x_n'^2,$$

where $x_n' = \ell$.

The discriminant of the form f is $\alpha_1\alpha_2\ldots\alpha_n$.

The discriminant of the form $f + \ell^2$ is $\alpha_1\alpha_2\ldots\alpha_{n-1}(\alpha_n+1)$. It exceeds the discriminant of the form f, since all the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$ are positive.

946 $$f(x_1, x_2, \ldots, x_n)$$
$$=a_{11}x_1^2+2a_{21}x_1x_2+\ldots+2a_{n1}x_1x_n+\varphi$$
$$=a_{11}\left(x_1+\frac{a_{21}}{a_{11}}x_2+\ldots+\frac{a_{n1}}{a_{11}}x_n\right)^2+f_1(x_2, \ldots, x_n),$$

where $f_1=\varphi-a_{11}\left(\frac{a_{21}}{a_{11}}x_2+\ldots+\frac{a_{n1}}{a_{11}}x_n\right)^2$.

The form f_1 is positive and its discriminant is D_f/a_{11}, where D_f is the discriminant of f. By problem 945, $D_{f_1}\geqslant\frac{D_f}{a_{11}}$, and the assertion is proved.

947 The proof is similar to that of Sylvester's law of inertia.

948 We construct the linear form

$$l_k = u_1 + u_2 x_k + \dots + u_n x_k^{n-1}, \quad k = 1, 2, \dots, n,$$

where $x_1, x_2, \cdots, x_n$ are roots of the given equation.

In this construction equal roots will correspond to equal forms, different roots to linearly independent ones, real roots to real forms, conjugate complex roots to conjugate complex forms.

The real and imaginary parts of a complex form $l_k = \lambda_k + \mu_k i$ are linearly independent of each other and of any forms that correspond to roots distinct from x_k, $\overline{x}_k$.

If we construct the quadratic form

$$f(u_1, \dots, u_n) = \sum_{k=1}^{n} (u_1 + u_2 x_i + \dots + u_n x_i^{n-1})^2$$

we have a form the rank of which is the number of distinct roots of the given equation. Its matrix of coefficients is

$$\begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{pmatrix}.$$

The sum of the squares of the complex conjugate linear forms $l_k = \lambda_k + i\mu_k$, $\overline{l_k} = \lambda_k - i\mu_k$ is $2\lambda_k^2 - 2\mu_k^2$. Therefore by the law of inertia, the number of negative terms in any canonical decomposition of f is equal to the number of

distinct pairs of conjugate complex roots of the given equation.

949 Follows from 948, 944.

950 The definition (f, φ) is obviously distributive. Thus it is sufficient to prove the assertion for squares of linear forms.

Let

$$f=(\alpha_1x_1+\alpha_2x_2+\dots+\alpha_nx_n)^2,$$
$$\varphi=(\beta_1x_1+\beta_2x_2+\dots+\beta_nx_n)^2.$$

It is easy to see that

$$(f,\ \varphi) = (\alpha_1\beta_1x+\alpha_2\beta_2x_2+\dots+\alpha_n\beta_nx_n)^2 \geqslant 0.$$

951 a) $4x_1'^2+x_2'^2-2x_3'^2$,
$$x_1'=\frac{2}{3}x_1-\frac{2}{3}x_2+\frac{1}{3}x_3,\quad x_2'=\frac{2}{3}x_1+\frac{1}{3}x_2-\frac{2}{3}x_3,\quad x_3'=\frac{1}{3}x_1+\frac{2}{3}x_2+\frac{2}{3}x_3;$$

b) $2x_1'^2-x_2'^2+5x_3'^2$,
$$x_1'=\frac{2}{3}x_1-\frac{1}{3}x_2-\frac{2}{3}x_3,\quad x_2'=\frac{2}{3}x_1+\frac{2}{3}x_2+\frac{1}{3}x_3,\quad x_3'=\frac{1}{3}x_1-\frac{2}{3}x_2+\frac{2}{3}x_3;$$

c) $7x_1'^2+4x_2'^2+x_3'^2$,
$$x_1'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3,\quad x_2'=\frac{2}{3}x_1+x_2+\frac{2}{3}x_3,\quad x_3'=-\frac{2}{3}x_1+\frac{2}{3}x_2+\frac{1}{3}x_3;$$

d) $10x_1'^2+x_2'^2+x_3'^2$,
$$x_1'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3,\quad x_2'=\frac{2\sqrt{5}}{5}x_1-\frac{\sqrt{5}}{5}x_2,\quad x_3'=\frac{2\sqrt{5}}{15}x_1+\frac{4\sqrt{15}}{15}x_2+\frac{\sqrt{5}}{3}x_3;$$

e) $-7x_1'^2+2x_2'^2+2x_3'^2$, $\quad x_1'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3,$

$$x_2'=\frac{2\sqrt{5}}{5}x_1-\frac{\sqrt{5}}{5}x_2,$$

$$x_3'=\frac{2\sqrt{5}}{15}x_1+\frac{4\sqrt{5}}{15}x_2+\frac{\sqrt{5}}{3}x_3;$$

f) $2x_1'^2+5x_2'^2+8x_3'^2$, $\quad x_1'=\frac{2}{3}x_1+\frac{1}{3}x_2+\frac{2}{3}x_3,$

$$x_2'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3,$$

$$x_3'=\frac{2}{3}x_1-\frac{2}{3}x_2-\frac{1}{3}x_3;$$

g) $7x_1'^2-2x_2'^2+7x_3'^2$, $\quad x_1'=\frac{\sqrt{2}}{2}x_1\quad-\frac{\sqrt{2}}{2}x_3,$

$$x_2'=\frac{2}{3}x_1+\frac{1}{3}x_2+\frac{2}{3}x_3,$$

$$x_3'=\frac{\sqrt{2}}{6}x_1-\frac{2\sqrt{2}}{3}x_2+\frac{\sqrt{2}}{6}x_3.$$

h) $11x_1'^2+5x_2^{2'}-x_3'^2$, $\quad x_1'=\frac{2}{3}x_1-\frac{2}{3}x_2-\frac{1}{3}x_3,$

$$x_2'=\frac{2}{3}x_1+x_2+\frac{2}{3}x_3,$$

$$x_3'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3;$$

i) $x_1'^2-x_2'^2+3x_3'^2+5x_4'^2$, $\quad x_1'=\frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}x_3+\frac{1}{2}x_4,$

$$x_2'=\frac{1}{2}x_1+\frac{1}{2}x_2-\frac{1}{2}x_3-\frac{1}{2}x_4,$$

$$x_3'=\frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_4,$$

$$x_4'=\frac{1}{2}x_1-\frac{1}{2}x_2-\frac{1}{2}x_3+\frac{1}{2}x_4;$$

j) $x_1'^2+x_2'^2-x_3'^2-x_4'^2$, $\quad x_1'=\frac{\sqrt{2}}{2}x_1+\frac{\sqrt{2}}{2}x_2,$

$$x_2'=\qquad\qquad\frac{\sqrt{2}}{2}x_3+\frac{\sqrt{2}}{2}x_4,$$

$$x_3'=\frac{\sqrt{2}}{2}x_1-\frac{\sqrt{2}}{2}x_2,$$

$$x_4'=\qquad\qquad\frac{\sqrt{2}}{2}x_3-\frac{\sqrt{2}}{2}x_4;$$

k) $x_1'^2+x_2'^2+3x_3'^2-x_4'^2$,
$$\begin{aligned}
x_1' &= \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_4,\\
x_2' &= \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_3,\\
x_3' &= \frac{1}{2}x_1+\frac{1}{2}x_2-\frac{1}{2}x_3-\frac{1}{2}x_4,\\
x_4' &= -\frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_4;
\end{aligned}$$

l) $x_1'^2+x_2'^2+x_3'^2-3x_4'^2$
$$\begin{aligned}
x_1' &= \frac{\sqrt{2}}{2}x_1+\frac{\sqrt{2}}{2}x_2,\\
x_2' &= \frac{\sqrt{2}}{2}x_3+\frac{\sqrt{2}}{2}x_4,\\
x_3' &= \frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_4,\\
x_4' &= \frac{1}{2}x_1-\frac{1}{2}x_2-\frac{1}{2}x_3+\frac{1}{2}x_4;
\end{aligned}$$

m) $x_1'^2-x_2'^2+7x_3'^2-3x_4'^2$,
$$\begin{aligned}
x_1' &= \frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}x_3+\frac{1}{2}x_4,\\
x_2' &= \frac{1}{2}x_1+\frac{1}{2}x_2-\frac{1}{2}x_3-\frac{1}{2}x_4,\\
x_3' &= \frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_4,\\
x_4' &= \frac{1}{2}x_1-\frac{1}{2}x_2-\frac{1}{2}x_3+\frac{1}{2}x_4;
\end{aligned}$$

n) $5x_2'^2-5x_2'^2+3x_3'^2-3x_4'^2$,
$$\begin{aligned}
x_1' &= \frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}x_3+\frac{1}{2}x_4,\\
x_2' &= \frac{1}{2}x_1+\frac{1}{2}x_2-\frac{1}{2}x_3-\frac{1}{2}x_4,\\
x_3' &= \frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_4,\\
x_4' &= \frac{1}{2}x_1-\frac{1}{2}x_2-\frac{1}{2}x_3+\frac{1}{2}x_4.
\end{aligned}$$

952 a) $\frac{n+1}{2}x_1'^2+\frac{1}{2}(x_2'^2+x_3'^2+\ldots+x_n'^2)$;

b) $\frac{n-1}{2}x_1'^2-\frac{1}{2}(x_2'^2+x_3'^2+\ldots+x_n'^2)$,

where
$$x_1' = \frac{1}{\sqrt{n}}(x_1+x_2+\ldots+x_n);$$
$$x_i' = \alpha_{i1}x_1+\alpha_{i2}x_2+\ldots+\alpha_{in}x_n, \quad i=2,\ldots,n,$$

where $(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$ is an arbitrary orthonormal basis for the solutions of the equation

$$x_i + x_2 + \ldots + x_n = 0.$$

953 $x_1'^2 \cos\frac{\pi}{n+1} + x_2'^2 \cos\frac{2\pi}{n+1} + \ldots + x_n'^2 \cos\frac{n\pi}{n+1}$.

954 If every proper value of the matrix A lies in the segment [a, b], then every proper value of the matrix A - λE is negative for λ > b and positive for λ < a. Therefore the quadratic form of matrix A - λE is negative definite for λ > b, and positive definite for λ < a. Conversely if the quadratic form (A - λE) X · X is negative definite for λ > b, and positive definite for λ < a, then every proper value for the matrix A - λE is positive for λ < a and negative for λ > b. Therefore for every proper value of the matrix A lies in the segment [a, b].

955 If X is an arbitrary vector, the following inequalities are satisfied:

$$aX \cdot X \leqslant AX \cdot X \leqslant cX \cdot X, \quad bX \cdot X \leqslant BX \cdot X \leqslant dX \cdot X.$$

Thus $(a+b)X \cdot X \leqslant (A+B)X \cdot X \leqslant (c+d)X \cdot X$.
Therefore every proper value of the matrix A + B lies in the segment [a+c, b+d].

956 a) Follows from the result of problem 937.
b) $|AX|^2 = AX \cdot AX = X \cdot A'AX \leq |X|^2 \cdot \|A\|^2$.

Equality can occur only if X is a proper vector of the matrix A'A corresponding to the proper value $\|A\|^2$.

c) Let X be an arbitrary vector; then

$$|(A+B)X| \leqslant |AX| + |BX| \leqslant (\|A\| + \|B\|)|X| .$$

But for arbitrary X_0,

$$|(A+B)X_0| = (\|A+B\|)\cdot|X_0|.$$

Therefore

$$\|A+B\| \leqslant \|A\| + \|B\|.$$

d) Let X be an arbitrary vector; then

$$|ABX| \leqslant \|A\|\cdot|BX| \leqslant \|A\|\cdot\|B\|\cdot|X|.$$

Now let X be the vector X_0 for which the relation $\|AB\|\cdot|X_0| = |ABX_0|$ holds, then $\|AB\| \leqslant \|A\|\cdot\|B\|$.

e) Let $\lambda = p + qi$ be a proper value of the matrix A; $X = Y + iZ$ be a proper vector that corresponds, then

$$AY = pY - qZ, \quad AZ = qY + pZ;$$

thus

$$|pY - qZ|^2 \leqslant \|A\|^2|Y|^2, \quad |qY + pZ|^2 \leqslant \|A\|^2|Z|^2.$$

Having these inequalities, we obtain

$$|\lambda|^2(|Y|^2+|Z|^2) = (p^2+q^2)(|Y|^2+|Z|^2) \leqslant \|A\|^2(|Y|^2+|Z|^2)$$

and therefore $|\lambda| \leqslant \|A\|$.

957 Let A be a real nonsingular matrix. Then $A'AX \cdot X = |AX|^2$ is a positive quadratic form that can be transformed into canonical form B by a succession of triagonal transformations with positive diagonal elements. Therefore $A'A = B'B$. Thus $(AB^{-1})' \cdot AB^{-1} = E$; $AB^{-1} = P$ is orthogonal. Thus $A = PB$. Uniqueness follows from the uniqueness of the triagonal transformation that carries a quadratic form into its canonical form.

958 $A'A$ is a symmetric matrix with positive proper values $\lambda_1, \cdots, \lambda_n$. Therefore

$$A'A = P^{-1}\mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}P.$$

Consider the matrix

$$B = P^{-1}\mathrm{diag}\{\mu_1, \mu_2, \cdots, \mu_n\}P,$$

where μ_i is the principal square root of λ_i. Clearly B is symmetric and has positive proper values; $B^2 = A'A$. Therefore $AB^{-1} = Q$ is orthogonal; $A = QB$.

959 A translation of the origin of a central surface to its center is such that if X is an arbitrary point, then after the translation either the surface contains both X, $-X$, or neither. In this way the first degree terms are eliminated from the equation of the surface.

The details are as follows. The transformation needed has the form $X = X_0 + X'$, where X_0 is the fixed translation vector. The equation of the surface becomes

$$AX' \cdot X' + 2(AX_0 + B)X' + AX_0 \cdot X_0 + 2BX_0 + C = 0.$$

A necessary and sufficient condition that a center exist is that the equation $AX_0 + B = 0$ have a solution X_0; and a necessary and sufficient condition for this is that the rank of the matrix A be equal to the rank of the matrix (A, B).

960 A preliminary translation of the origin to the center of the surface transforms the equation into

the form $AX \cdot X + \gamma = 0$.

If r is the rank of A and $\alpha_1, \cdots, \alpha_r$ are the non-zero proper values of A, then a further orthogonal transformation will transform the equation of the surface into the canonical form $\alpha_1 x_1^2 + \ldots + \alpha_r x_r^2 + \gamma = 0$.

961 The surface will not have a center if the rank of (A, B) exceeds the rank of A. This can only occur if R = rank $A < n$. Let R be the entire space; let P be the space AR, let Q be the orthogonal complement of P. Then if Y is an arbitrary vector in Q, the relation $AY = 0$ must hold. Thus $|AY|^2 = AY \cdot AY = Y \cdot AAY = 0$, since $AAY \in P$. Let B have the decomposition $B = B_1 + B_2$, $B_1 \in P$, $B_2 \in Q$. Then $B_2 \neq 0$; otherwise B would lie in P, and rank (A, B) would be r. Set $B_1 = AX_0$. In other words let X_0 be the inverse image of B_1. The translation $X = X_0 + X'$ transforms the equation of the surface into

$$AX' \cdot X' + 2B_2X' + c' = 0.$$

We now make the further translation $X' = aB_2 + X''$. Then $AX' \cdot X' = a^2AB_2 \cdot B_2 + 2aAB_2 \cdot X'' + AX'' \cdot X'' = AX'' \cdot X''$. But $AB_2 = 0$; thus the equation is finally transformed into

$$AX'' \cdot X'' + 2B_2X'' + 2a\,|B_2|^2 + c' = 0.$$

If we choose $a = -\frac{1}{2}c/|B_2|^2$, the equation becomes

$$AX'' \cdot X'' + 2B_2X'' = 0.$$

Now we make an orthogonal coordinate transformation to a basis of orthogonal unit proper vectors of the matrix A, taking one of these vectors to be the unit vector collinear with B_2 but oppositely directed. This is possible since B_2 is a proper vector of A. In terms of the new basis, the equation has the form

$$\lambda_1 x_1^2 + \dots + \lambda_r x_r^2 - 2\beta_2 x_{r+1} = 0,$$

where $\beta_2 = |B_2|$. Finally we divide by β_2.

962 The matrix A induces a linear transformation of the space into the subspace spanned by the vectors $A_1, A_2, \cdots, A_n$, the latter being the column vectors of A. The assertion follows directly from this.

963 Let $e_1, e_2, \dots, e_q$ be a basis of Q; then Q' is the space spanned by $e'_1, e'_2, \dots, e'_q$, where $e'_1, e'_2, \dots, e'_q$ are the images of $e_1, e_2, \dots, e_q$ under a certain linear transformation. Therefore $q' \le q$. Moreover it is clear that $q' \le r$ since Q' is contained in R'. Now let P be the space complementary to Q; the dimension of P is $p = n - q$. Let P' be the image of P under the linear transformation above. The dimension of the latter, p', will not exceed $n - q$. But $P' + Q' = R'$; therefore $p' + q' \ge r$. Therefore $q' \ge r - p' \ge r + q - n$.

964 Let A have rank r_1; B have rank r_2; $BR = Q$. The dimension of Q is r_2. Then the rank ρ of

AB is equal to the dimension of $ABR = AQ$. The preceding problem shows that $r_1 + r_2 - n \leq \rho \leq \min(r_1, r_2)$.

965 Any projection operator is idempotent; i.e. $A^2 = A$. To see this, suppose that A projects the entire space R on the subspace P; then a repetition of this transformation does not decrease P. Therefore $A^2 = A$. Conversely suppose $A^2 = A$. Let P be the space of all vectors of the form $Z = AX$; let Q be the space of all vectors Y such that $AY = 0$. It is clear that P, Q are linear spaces. Their intersection is null since the two relations $AX = Y$, $AY = 0$ lead to $AX = A^2X = AY = 0$. Thus if X is any vector, $X = AX + (E - A)X$. On the one hand, $(E - A)X \in Q$, so that $A(E - A)X = (A - A^2)X = 0$. On the other hand X can be written as the sum of an element in Q and an element in P (see above). Thus P + Q is the entire space; P, Q are complementary subspaces. The operator P carries the arbitrary vector X into its P-component. Thus A is a projection onto P parallel to Q.

966 Let P, Q be orthogonal. We combine orthonormal bases of P, Q to form an orthonormal basis of the entire space. In terms of this basis the matrix of A has the diagonal form

$$\tilde{A} = \mathrm{diag}\{1, 1, \cdots, 1, 0, 0, \cdots, 0\},$$

where all the ones or all the zeros may be missing. If any other orthonormal basis is chosen, the projection will have the matrix $A = B^{-1}\tilde{A}B$, where

B is some orthogonal matrix. It is clear that A is symmetric.

Conversely, let A be an arbitrary symmetric matrix, with $A^2 = A$. The spaces $P = AR$, $Q = (E - A)R$ are clearly orthogonal; and the relations

$AX \cdot (E - A)Y = X \cdot A'(E - A)Y = X \cdot (A - A^2)Y = 0$

hold.

967 Let A be a skew-symmetric matrix. We first notice that if X is an arbitrary vector, the relation $AX \cdot X = 0$ holds. Indeed we have

$$AX \cdot X = X \cdot A'X = X \cdot (-AX) = -AX \cdot X.$$

Let $\lambda = \alpha + \beta i$ be a proper value of the matrix A; let $U = X + Yi$ be the corresponding proper vector. Then

$$AX = X\alpha - Y\beta, \quad AY = X\beta - Y\alpha.$$

From this we conclude that $\alpha(|X|^2 + |Y|^2) = AX \cdot X + AY \cdot Y = 0$; $\alpha = 0$. Further, $\beta X \cdot Y + \alpha |Y|^2 = AY \cdot Y = 0$; thus $X \cdot Y = 0$ if $\beta \neq 0$. Finally we see that $|X| = |Y|$ from the relations $\beta(|X|^2 - |Y|^2) = AY \cdot X + AX \cdot Y = AY \cdot X - X \cdot AY = 0$.

968 Let the skew-symmetric matrix A have the form

$$A = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{12} & 0 & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{1n} & -a_{2n} & \dots & 0 \end{pmatrix}$$

We first remark that if every proper value of A is 0, then A = 0. To see this note that the sum of the products of the proper values two by two is the sum of the second order principal minors $\sum_{i<k} a_{ik}^2$. This can be 0 only if every element a_{ik} is 0.

Now let $\lambda_1 = a_1 i$ be an arbitrary non-zero proper value of A. Let X, Y be the real and imaginary parts of the corresponding proper vector. By problem 967 these have the same length. Thus we can norm them simultaneously by dividing the vector X + Yi by an appropriate scalar factor. We then have the relations

$$AX = -Ya_1; \quad AY = Xa_1.$$

Using the Gram-Schmidt orthogonalization process we can construct an orthogonal matrix B having X, Y for its first two columns. Then the matrix $P^{-1}AP$ has the form

$$P^{-1}AP = \begin{pmatrix} 0 & a_1 & \dots \\ -a_1 & 0 & \dots \\ 0 & 0 & \dots \\ \cdot & \cdot & \cdot \\ 0 & 0 & \dots \end{pmatrix}.$$

But this matrix must be skew-symmetric so that the elements in the first two rows not expressly written must all be zero. An inductive argument similar to that used to prove Schur's

theorem (see the remarks after problem 931) shows that every skew-symmetric matrix can be transformed into the form

$$\operatorname{diag}\left\{\begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a_2 \\ -a_2 & 0 \end{bmatrix}, \cdots\right\}$$

by an orthogonal transformation.

969 Suppose

$$B = (E - A)(E + A)^{-1}$$

Then

$$B' = \{(E - A)^{-1}\}' (E - A)' = (E - A)^{-1}(E + A) = B^{-1}.$$

Thus

$$B + E = (E - A)(E + A)^{-1} + (E + A)(E + A)^{-1} = 2(E + A)^{-1},$$

and therefore

$$\det[B + E] \neq 0.$$

Conversely, suppose

$$B' = B^{-1}, \ \det[B + E] \neq 0.$$

Then we can take the matrix $(E + B)^{-1}(E - B)$ as a possible matrix A. It is easy to see that A is skew-symmetric.

970 Let A be an arbitrary orthogonal matrix. Then if X, Y are arbitrary vectors, we have

$$AX \cdot AY = X \cdot A'AY = X \cdot Y$$

Let

$$\lambda = \alpha + \beta i$$

be a proper value of the matrix A; let

$$U = X + Yi$$

be the corresponding proper vector. Then

$$AX = X\alpha - Y\beta, \qquad AY = Y\alpha + X\beta.$$

Thus

$$|X|^2 = X \cdot X = AX \cdot AX = \alpha^2|X|^2 + \beta^2|Y|^2 - (X2\alpha\beta) \cdot Y,$$
$$|Y|^2 = Y \cdot Y = AY \cdot AY = \alpha^2|Y|^2 + \beta^2|X|^2 + (X2\alpha\beta) \cdot Y.$$

We obtain the relation $\alpha^2 + \beta^2 = 1$ by adding the last two equations.

971 If $\beta \neq 0$, we can subtract the two equations above and obtain the relation

$$\beta(|X|^2 - |Y|^2) + (X2\alpha) \cdot Y = 0.$$

On the other hand

$$X \cdot Y = AX \cdot AY = (\alpha^2 - \beta^2) X \cdot Y + \alpha\beta(|X|^2 - |Y|^2).$$

Thus

$$\alpha(|X|^2 - |Y|^2) - 2\beta X \cdot Y = 0.$$

And finally,

$$X \cdot Y = |X|^2 - |Y|^2 = 0.$$

972 1. Let $\lambda = \alpha + \beta i = \cos\varphi + i \sin\varphi$ be a non-real proper value of the matrix. We construct an orthogonal matrix Q having for its first two columns the real and imaginary parts of the proper vector that corresponds to λ. Then

$$Q^{-1}AQ = \begin{pmatrix} \cos\varphi & \sin\varphi & \dots \\ -\sin\varphi & \cos\varphi & \dots \\ 0 & 0 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \\ 0 & 0 & \dots \end{pmatrix}.$$

Since the matrix $Q^{-1}AQ$ is orthogonal, the sum of the squares of the elements of each row is 1. Therefore all the elements in the first two rows not explicitly written down must be 0.

2. Let $\lambda = \pm 1$ be a real characteristic number of A; let X be the normalized proper vector corresponding to λ.

We construct an orthogonal matrix Q, the first column of which is the vector X. Then

$$Q^{-1}AQ = \begin{pmatrix} \pm 1 & \dots \\ 0 & \dots \\ \cdot & \cdot\ \cdot\ \cdot \\ 0 & \dots \end{pmatrix}.$$

Using the same argument as before we see that all the elements in the first row, except the first, are 0. The proof is completed by induction (see the argument following problem 931).

973 a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$; f) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; k) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$;

b) $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; g) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; l) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$;

c) $\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; h) $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; m) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$;

d) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$; i) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; n) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

e) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$; j) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; o) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

974 a) $\begin{bmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 1 & 1 \\ & & 0 & 1 \end{bmatrix}$; b) $\begin{bmatrix} 1 & 1 & & \\ 0 & 1 & 1 & \\ & 0 & 1 & 1 \\ & & 0 & 1 \end{bmatrix}$;

c) $\begin{bmatrix} \varepsilon & & & & \\ & \varepsilon^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \varepsilon^{n-1} \end{bmatrix}$,

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

975 The box

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & & \lambda & 1 \end{bmatrix}$$

of dimension > 1 cannot have finite order.

Remark. The result of this problem is not valid over a field of nonzero characteristic. For example, if the characteristic of the field is 2, the relation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = E$ holds.

976 Let A be the given matrix and $B = C^{-1}AC$ be its Jordan canonical form. The canonical matrix B

is triagonal and its diagonal elements are the proper values of the matrix A; each one appears on the diagonal as many times as its multiplicity in the characteristic polynomial. Thus $B'_m = (C'_m)^{-1} A'_m C'_m$ (see problem 531). Therefore the matrices A'_m, B'_m have the same characteristic polynomials. If the rows and columns of B'_m are properly numbered it has triagonal form and therefore its proper values are its diagonal elements; these are obviously the products of the proper values of A taken m at a time.

977 It is obvious that the matrices $A - \lambda E$, $A' - \lambda E$ have the same elementary divisors. Therefore A, A' can be transformed into the same canonical form and therefore one can be transformed into the other.

978 Set A = CD, where C, D are symmetric matrices and C is nonsingular. Then A' = DC; therefore $A' = C^{-1}AC$. Therefore the matrix C is a matrix that transforms A into A'.

Set $A = SBS^{-1}$, where B is a canonical matrix:

$$B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}, \quad \text{where} \quad B_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix}.$$

Then $A' = S'^{-1}B'S'$. Let H_i be the matrix

$$H_i = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

It is clear that $B_i' = H_i^{-1}B_iH_i$. Therefore

$B' = H^{-1}BH$, where $H = \mathrm{diag}\{H_1, H_2, \cdots, H_k\}$.

Thus $A' = S'^{-1}H^{-1}BHS' = S'^{-1}H^{-1}S^{-1}ASHS' = C^{-1}AC$, where $C = SHS'$. The matrix C is obviously symmetric. Set $D = C^{-1}A$, then $D' = A'C^{-1} = C^{-1}ACC^{-1} = D$. Thus the matrix D is also symmetric; $A = CD$.

979 Let $\det[A - \lambda E] = (-1)^n(\lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - \ldots - c_n)$. First note that $p_1 = \mathrm{Sp}\, A = c_1$. As an induction hypothesis suppose that $p_1 = c_1,\ p_2 = c_2,\ \ldots,\ p_{k-1} = c_{k-1}$, and let us establish the relation $p_k = c_k$. By construction $A_k = A^k - p_1A^{k-1} - p_2A^{k-2} - \ldots - p_{k-1}A$ $= A^k - c_1A^{k-1} - c_2A^{k-2} - \ldots - c_{k-1}A$. Therefore

$$\begin{aligned}\mathrm{Sp}A_k = kp_k &= \mathrm{Sp}A^k - c_1\mathrm{Sp}A^{k-1} - \ldots - c_{k-1}\mathrm{Sp}A \\ &= S_k - c_1S_{k-1} - \ldots - c_{k-1}S_1,\end{aligned}$$

where $S_1, S_2, \cdots, S_k$ are power sums of the proper values of the matrix A. But by Newton's formula $S_k - c_1S_{k-1} - \ldots - c_{k-1}S_1 = kc_k$. Therefore $p_k = c_k$.

Now by the Hamilton-Cayley theorem,
$B_n = A^n - c_1A^{n-1} - \dots - c_{n-1}A - c_nE = 0$. Finally
$AB_{n-1} = A_n = c_nE$, so that $B_{n-1} = c_nA^{-1}$.

980 If C has the form

$$C = \begin{pmatrix} 0 & c_{12} & \dots & c_{1n} \\ c_{21} & 0 & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & 0 \end{pmatrix}$$, with zero diagonal,

the assertion is clear. Indeed we can take

$$X = \operatorname{diag}\{\alpha_1,\ \alpha_2, \cdots, \alpha_n\},$$

with arbitrary distinct diagonal elements. Then we can set

$$Y = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$$, and note that

$$XY - YX = \begin{pmatrix} 0 & b_{12}(\alpha_2 - \alpha_1) & \dots & b_{1n}(\alpha_n - \alpha_1) \\ b_{21}(\alpha_1 - \alpha_2) & 0 & \dots & b_{2n}(\alpha_n - \alpha_2) \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1}(\alpha_1 - \alpha_n) & b_{n2}(\alpha_2 - \alpha_n) & \dots & 0 \end{pmatrix}$$

The problem is solved by taking

$b_{ik} = c_{ik}/(\alpha_k - \alpha_i)$; $i \neq k$.

To finish, it is sufficient to show that every matrix C with zero trace is similar to a matrix with zero diagonal. This we prove by induction as follows. If $\mathrm{Sp}\, C = 0$, then C cannot have the form $C = \mu E$ unless $\mu = 0$. Therefore there must be a vector U such that U, CU are independent. Let W be a nonsingular matrix having U, CU for its first two columns. Then

$$W^{-1}CW = C' = \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1n} \\ 1 & \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{bmatrix} O & P \\ Q & M \end{bmatrix}.$$

Therefore, $\mathrm{Sp}\, M = \mathrm{Sp}\, C = 0$.

By an appropriate induction hypothesis we have $M = S^{-1}M'S$, where M' is a matrix with zero diagonal. Thus the matrix

$$C'' = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}^{-1} C' \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} = \begin{bmatrix} 0 & PS \\ S^{-1}Q & S^{-1}MS \end{bmatrix}$$

has zero diagonal.

981 The series $H = I + CC^* + C^2 C^{*2} + \cdots$ converges, since $||C|| < 1$ (see problem 956). Direct substitution shows that H satisfies the relation given.

See P. Stein, J. Research Nat. Bur. Stds. 48 (1952) 82-83.

982 Similar to 981; $||C^k BC^{*k}|| \leq ||B|| \cdot ||C||^{2k}$.

983 Set $K = I - A$, $L = I + A$; note that K^{-1} exists. Define C by $C = LK^{-1} = K^{-1}L = (I + A)(I - A)^{-1} = (I - A)^{-1}(I + A)$. By problem 931, all proper values of C lie in the unit circle, since the quantity $(1 + \rho)(1 - \rho)^{-1} = (1 + x + iy)(1 - x - iy)$ has modulus less than 1 if x is a negative number.

We are to assume that there is a positive-definite hermitian matrix G such that

$$(2) \qquad G - CGC^* = K^{-1}(2R)(K^{-1})^*,$$

the right member of the latter equation being positive-definite hermitian, since R is.

We write (2) in the form

$$G - K^{-1}LGL^* K^{*-1} = K^{-1}(2R)\, K^{*-1}, \text{ or}$$

$$KGK^* - LGL^* = 2R.$$

Direct substitution yields

$$KGK^* = (I - A)G(I - A^*) = G - GA^* - AG + AGA^*,$$

$$-LGL^* = -(I + A)G(I + A^*) = -G - GA^* - AG - AGA^*,$$

which establishes the assertion.

See O. Taussky, SIAM Journal 9 (1961), 640-643, and many later papers.

Faddeev / Sominskii

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